STABILITY OF LARGE-SCALE SYSTEMS

— Well-Posedness, Frequency Domain Criteria, and Estimate of Stability Region —

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ABSTRACT

As technological systems continuously grow in size and complexity, we need a large-scale system theory to overcome the computational and other sorts of difficulities by the increase of the size of the systems.

In this thesis, several problems which concern with the Lyapunov stability of large-scale systems are studied. The focus is set upon two special classes of large-scale systems. The well-posedness, stability conditions, and the stability regions of such systems are studied.

In Chapter 1, general results concerning the stability of composite systems are reviewed, and the significance of this thesis is clarified. Further, a little more detailed reviews on the past researches about the well-posedness conditions, stability conditions and stability regions are made.

In Chapter 2, the well-posedness of large-scale systems described by functional relations is studied, and a sufficient condition for well-posedness is obtained.

In Chapter 3, a composite system method in the stability analysis of large-scale systems which are described by differential equations is studied. A stability condition and an estimate of the stability region are given. The former result is used in Chapter 5.

In Chapter 4, the Lyapunov stability of a multi-input multi-output Lur'e type system with time-varying nonlinearities is studied. A frequency domain condition containing an arbitrary parameter "weight" is obtained, and the relation between this weighted multivariable circle criterion and the other two conditions are studied. A method of searching a feasible weight is proposed.

In Chapter 5, the Lyapunov stability of composite systems which are composed of Lur'e type subsystems is studied. Four cases of the composite systems are considered according to the subsystem assumptions. In each case, a frequency domain condition is obtained. And the relation between these conditions and L_2-stability conditions are clarified.

In Chapter 6, the results of Chapters 4 and 5 are summarized from a viewpoint of constructing a Lyapunov function and the estimates of the stability

regions are given by using these Lyapunov functions.

In Chapter 7, eight examples are examined to compare the results of the preceding chapters with other conditions. As a practical example, the transient stability of a multi-machine power system is studied.

In Chapter 8, the results of this thesis are summarized.

GLOSSARY

Notations

$\underline{a}$, $\underline{a}_j$	; vectors (underscored)
a_j	; j-th component of the vector $\underline{a}$
$a_j^{(k)}$	; j-th component of the vector $\underline{a}_k$
A, A_j, A_{jk}	; matrices
$[A_\ell]_{jk}$, $a_{jk}^{(\ell)}$	; jk-th element of the matrix A_ℓ
I	; unit matrix
$A > B$	; the matrix $A - B$ is positive-definite
$A \geq B$	; the matrix $A - B$ is positive-semi-definite
A^*	; conjugate transpose of the matrix A
A^T	; transpose of the matrix A
$\lambda_{max}(A)$	; maximum eigenvalue of the matrix A if it exists
$\lvert a \rvert$	; absolute value of the scalor a
$\lvert \underline{a} \rvert$	; a vector whose j-th component is $\lvert a_j \rvert$
$\lVert \underline{a} \rVert$	; Euclidean norm of the vector $\underline{a}$
$\bullet$	; d/dt

In Chapter 2, we use the symbols A, A^*, $\lVert \underline{a} \rVert$ in a different sense.

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Chapter 1 Introduction

In the last few decades technological systems have been continuously growing in size and in complexity. To catch up with this situation, we need new systematic methods to overcome the computational and other sorts of difficulties by the increase of the size of the systems. Actually, many researchers have been trying to establish a large-scale system theory (Mesarovic & Takahara 1975, Michel & Miller 1977, Šiljak 1978, IEEE Control society 1978). This thesis also aims at making a contribution in this field. Especially, we study the problems which concern with the Lyapunov stability of large-scale systems. The focus is set upon two special classes of the large-scale systems which often appear in engineering problems. The well-posedness condition, the stability condition and the stability regions of such systems are studied.

The fundamental method used in this thesis is to regard a large-scale system as the interconnection of smaller systems (i.e. subsystems) and to make three step analysis. Namely, we first analyze the subsystem independently, next analyze the interconnection, and lastly combine the results to reduce the property of the whole. This method has been called the "composite system method" or "decomposition-aggregation method". The use of the composite system method in the stability analysis of control systems appeared first in Bailey's paper (1966) whereas the basic ideas trace back to Bellman (1962) and Matrosof (1962). Up to now, the composite system method has been developed both for the Lyapunov stability analysis and the input-output stability analysis. As for the former, the method can be classified into two sub-classes, i.e. the vector Lyapunov function method and the scalar Lyapunov function method. The principal idea of the vector Lyapunov function method is to use a set of functions which respectively represent the subsystems and to make a comparison system which reflects the stability of the original one. This method has been studied by Baily (1966), Šiljak (1972, 1978), Grujic et al. (1976), Michel (1974), Weissenberger (1973), and Mori (1977). On the other hand, the principal idea of the scalar Lyapunov function method is to construct a Lyapunov function for each subsystem and to use the weighted sum of these Lyapunov functions as the candidate for the Lyapunov function of the whole system. This method has been

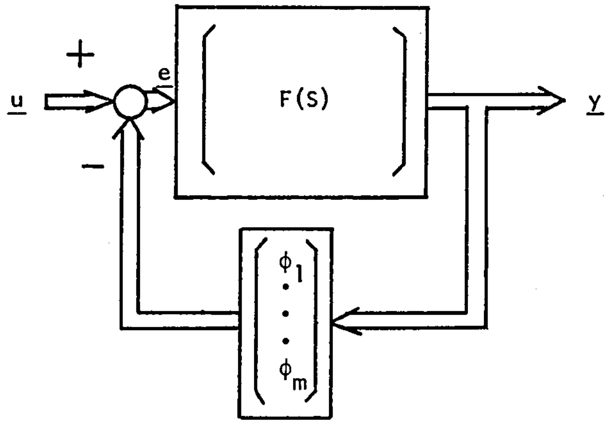

Fig. 1-1   System with multiple nonlinear feedbacks
F(s) is a mxm transfer function matrix.

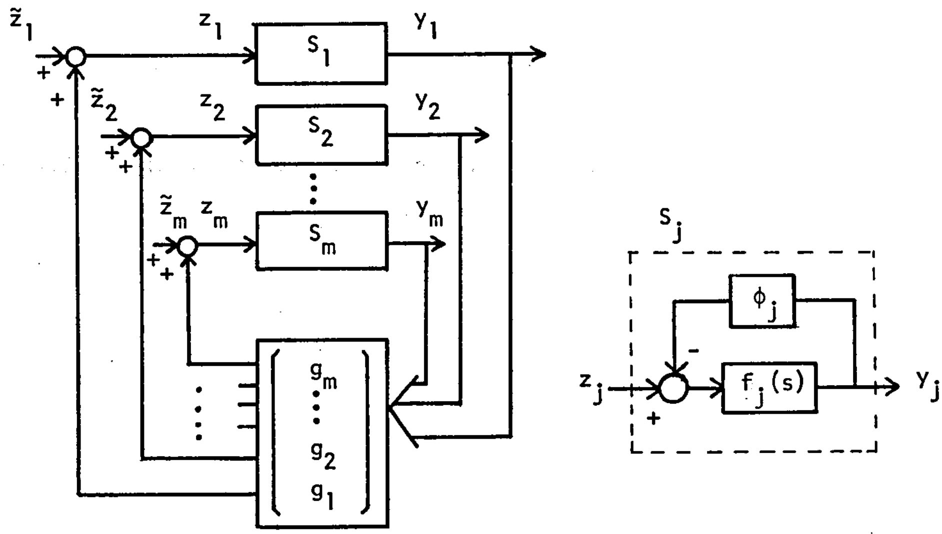

Fig. 1-2 Composite systems consisting of subsystems with nonlinear feedbacks

studied by Michel (1970, 1975), Thompson (1970), Araki (1971), Araki & Kondo (1972) Michel & Porter (1972), Suda (1973) and Šiljak (1973). The comparison of the above two methods are as follows. Bailey's condition (1966), which uses the vector Lyapunov function method, is included in the condition obtained by the scalar Lyapunov function method (Thompson 1970, Araki 1971, Araki & Kondo 1972, Michel & Porter 1972). Generally, stability conditions which can be obtained by the vector Lyapunov function method with linear, time-invariant comparison systems are always reduced by the scalar Lyapunov function method (Araki 1978b). Application of the composite system method to the input-output stability was studied by Tokumaru et al. (1974), Lasley & Michel (1976), and Araki (1976a). As the L_2-stability, which is a specific class of the input-output stability, implies the asymptotic stability in the large, the L_2-stability condition is also useful for the analysis of the Lyapunov stability. The input-output analysis has advantage in giving a transparent outlook, and actually simple frequency domain conditions are obtained for the systems composed of Lur'e type subsystems (Tokumaru et al 1974, Lasley & Michel 1976, Araki 1976). However, this method has an important drawback that it cannot deal with local stability directly and is importent in evaluating the stability regions. Since, usually, practical large-scale systems can be stable only locally and since the stability region is an important factor to evaluate the quality of the large-scale systems, the L_2-stability analysis alone cannot be a satisfactory tool for our purpose.

Now, a number of general results are available concerning the stability of composite systems as reviewed in the above. However, not much have been done about the application of those conditions to specific systems. Especially, construction of subsystems' Lyapunov functions has been the bottleneck in the application and only a few results are available up to now. In this thesis, we focus on the two classes of systems shown in Fig. 1.1 and Fig. 1.2 and make an intensive study about their Lyapunov stability and related problems. In deriving the Lyapunov stability condition of such systems, we can think of two possible ways as mentioned in the last paragraph; one is to apply the composite type theorem on the Lyapunov stability directly, and the other is to use the composite type theorem on the L_2-stability and assure the Lyapunov stability in the large by virtue of the general theorem on the relation of L_2-stability and Lyapunov stability. If we follow these two ways and obtain stability conditions for the system of Fig. 1.2, the two results do not coincide. To be more accurate the conditions obtained by the direct application of the Lyapunov stability theorem turn out to be more conservative than the conditions obtained by the L_2-stability theorem. This fact suggests that the existing theorem on the Lyapunov stability of composite systems is not satisfactory with respect to its sharpness. In the following, we first improve the Lyapunov stability theorem on the composite systems so that sharper results can be obtained. Then, we present a method to construct a Lyapunov function of the systems of Fig. 1.1 and 1.2 based on the positive-real lemmas. As the results of these schemes we can directly derive the Lyapunov stability condition which agrees with the L_2-stability conditions (Araki 1976, Saeki et al. 1979). We also study a method of estimating the stability regions using the above Lyapunov functions. In applying the above results to practical (physical) systems there is a fundamental question whether the basic differential equation represents the physical system appropriately or not. This question arises when more than two mathematical models are combined to describe a larger system, which is generally called the well-posedness problem. Since our basic idea is to describe a large-scale system as the interconnection of smaller parts, this well-posedness problem is of primary importance for the application of our results. So, we use one chapter for the study of the well-posedness of composite systems.

Now, let us make a little more detailed review on the past researches about the well-posedness conditions, the stability conditions and the stability regions which can be applied to the classes of the large-scale systems treated in this thesis.

First, let us review the researches about the well-posedness conditions. Zames (1964), J.C. Willems (1971b), and Desoer & Vidyasagar (1975) studied this problem for single-loop feedback systems. Concerning large-scale systems, Ikeda & Kodama (1973), and Vittaco & Michel (1977) gave sufficient conditions for the existence and uniqueness of the solution, where the former dealt with the systems described by a block diagram form and considered the existence of a state space representation at the same time, while the latter treated systems described by a set of differential and algebraic equations and studied the stability problem at once. Maeda & Kodama (1977) and Vidyasagar (1980) studied a stricter well-posedness property which requires the existence of the solution also for the systems obtained by inserting parasitic delay elements. His result has advantage in the points that he employed the functional description of systems which includes the above two types of descriptions, that the requirements of the well-posedness are stronger, and that the condition is given in terms of the properties of the subsystems and interconnections. But it contains the drawback that the condition does not reduce to Willems's condition (Th. 4.1 1971a) when applied to the single-loop feedback system. This is mainly because Vidyasagar's condition is concerned with the "qualitative" (i.e. graph-theoretical) nature of the systems and does not take the "quantitative" property into consideration. We will study this topic in Capter 2. We present a quantitave condition which reduces to the Willems's condition when applied to a single-loop system and which is expressed in terms of the instantaneous gains of subsystems and interconnections. Our result includes the Vidyasagar's condition, and is more powerful than his in the point that we can guarantee the well-posedness of such systems which contain loops without any smoothing operator.

Secondly, let us review the researches about the stability of the system given in Fig. 1.1. This system has been studied in the absolute stability problem by many researchers (Popov 1973; Narendra & Taylor 1973). The single-input single-output case was studied first, and the result was extended to the multi-input multi-output case. It was shown that a certain class of Lyapunov function exists if and only if a frequency domain condition is satisfied. This result gives a method of constructing a Lyapunov function using the frequency domain condition. This frequency domain condition is easy to apply except for the next two points. One point is that the condition contains arbitrary parameters and do not have any practical means to choose them. The other point is that, because of its matrix nature, it is not as convenient for graphical use as in the scalar condition. As far as the author knows, there is no research about the former. As for the latter, some graphical conditions, which were the sufficient conditions of the frequency domain condition, were obtained (Rosenbrock 1973, Cook 1973, Shankar & Atherton 1977, Mees & Atherton 1980). On the other hand, by applying the composite system method , a simple criterion using an M-matrix[†] was obtained for L_2-stability of the above system (Araki 1976a). This new criterion bears strong resemblance to the above mentioned graphical conditions by Rosenbrock and by Cook. This fact suggests that there exists a strong connection between the M-matrix condition and the frequency domain condition. We will study this topic in Chapter 4. We assume that the nonlinearities ϕ_j are time-varying. The well-posedness condition of the system is obtained using the result of Chapter 2. A frequency domain condition which contains arbitrary parameter " weight " is obtained and this condition is referred to as the "weighted multivariable circle criterion." The relation between the weighted multivariable circle criterion and the L_2-stability criterion, and also the Rosenbrock's criteria are studied. A method of searching a feasible value of weight is proposed.

Thirdly, let us review the researches about the stability conditions of the system given in Fig. 1.2. Araki (1971) and Šiljak (1972) applied their Lyapunov stability conditions of composite systems to this system, respectively. The obtained conditions contain parameters of Lyapunov functions used in their proof. As L_2-stability conditions do not contain these parameters, it is obviously difficult to clarify the relation between these Lyapunov stability conditions and the L_2-stability conditions. In the special cases of the system, the relation between the Lyapunov stability and the input-output stability was studied. Namely, Cook (1974) considered the case that ϕ_j's are absent and g_j's are linear time invariant, and Araki (1978a) considered the case that ϕ_j's satisfy infinite sector conditions and g_j's are linearly bounded nonlinearities. They used a quadratic order Lyapunov function and showed the equivalent relation between the Lyapunov stability and the L_2-stability for these systems. We will study this topic in Chapter 5. The system of Fig. 1.2 is classified into four classes and is described precisely. It is assumed that Φ_j's satisfy a finite sector condition and g_j's are linearly bounded nonlinearities. The system is shown to

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†) The definition of M-matrices and their fundamental properties are given in Appendix.

be well-posed. Circle criterion type and Popov criterion type conditions are obtained using the new criterion given in Capter 3. These conditions assure the existence of Lyapunov functions of the type " quadratic form " and " quadratic form plus the integral of the nonlinear function " for each subsystem, respectively. The relations between these conditions and the L_2-stability conditions (Araki 1976a, Saeki & Araki 1979) are clarified.

Lastly, let us review the researches about the estimate of the stability regions. For a single-input single-output Lur'e type system (the system of Fig. 1.1 with m=1), the estimate of the stability region was obtained on condition that the system description is valid only for some bounded values of the output (Walker & McClamroch 1967, Noldus et al.1973, J.L. Willems 1969). This result was extended to a multi-input multi-output case (Pai & Narayana 1976). In these studies the linear part of the Lur'e type system was assumed to have no feed-through. Concerning composite systems, to obtain the estimate of the stability region using the knowledge of subsystems was considered by Weissenberger (1973) and Bitsoris & Burgat (1976). They assumed that each subsystem has a first order Lyapunov function and derived their results based on the stability condition of Šiljak (1972). It is probable that we can obtain a larger estimate of the stability region if we use some other conditions sharper than Šiljak's condition. We will study this topic in Chapters 3 and 6. In Chapter 3, the stability region of general composite systems is estimated using the new stability condition given in this chapter. This new condition is sharper than the Šiljak's condition. In Chapter 6, we obtain the estimates of the stability regions of the systems given in Figs. 1.1 and 1.2. First, we presented a concrete method of constructing Lyapunov functions of these systems. This method is directly obtained from the proof of the stability conditions of Chapters 4 and 5. Next we obtain the estimates of the stability regions. As we assume that the linear part of the system of Fig. 1.1 has feed-through, the derivation of the stability region becomes rather complicated compared with the previous studies. The stability region estimate of the system given in Fig. 1.2 is obtained by applying the results of Chapter 3 and the results of the stability region estimate of Lur'e type systems.

Chapter 2 Well-Posedness Condition of Large-Scale Systems

In this chapter, we study the well-posedness of large-scale systems. We deal with the class of systems which are described by functional relations. The systems described by differential equations are a subclass of the systems considered in this chapter. The result of this chapter can be easily transformed to fit the differential equation description.

Sec. 2.1. Preliminary Study

Let us study the notation and terminology which are used in this chapter.

The S denotes the interval $[T_0, \infty)$. The p is a positive number greater than or equal to 1 and is fixed throughout this chapter. For a q-vector $\underline{\xi}$, $||\underline{\xi}||$ denotes the p-th norm, i.e.

$$||\underline{\xi}|| = \{ \sum_{\ell=1}^{q} |\xi_\ell|^p \}^{1/p}$$

where ξ_ℓ is the ℓ-th component of $\underline{\xi}$. L_p^q is the linear space of the p-th power intergrable R^q-valued functions defined on S, and $||\underline{x}||$ denotes its norm; i.e.

$$L_p^q = \{ \underline{x} \mid \int_{T_0}^{\infty} ||\underline{x}(t)||^p \, dt < \infty \}, \quad ||\underline{x}|| = [\int_{T_0}^{\infty} ||\underline{x}(t)||^p \, dt]^{1/p}$$

For a qxq' matrix F, $||F||$ denotes the norm induced from the above defined vector norm; i.e. $||F|| = \max_{||\underline{\xi}||=1} ||F\underline{\xi}||$. For $T \in S$, P_T denotes the truncation operator; i.e. for a function defined on S, $P_T\underline{x}$ is given by

$$(P_T\underline{x})(t) = \begin{cases} \underline{x}(t) & \text{for } T \geqq t \geqq T_0 \\ \underline{0} & \text{for } \quad t > T \end{cases}$$

The L_{pe}^q is the extension of L_p^q ; i.e.

$$L_{pe}^q = \{ \underline{x} \mid P_T\underline{x} \in L_p^q \text{ for any } T \in S \}$$

The equality of $\underline{x}$ and $\underline{x}'$ in L_{pe}^q is defined by $P_T\underline{x} = P_T\underline{x}'$ in L_p^q for any $T \in S$. For $\varepsilon > 0$, D_ε denotes the *delay operator* on L_{pe}^q, i.e. $D_\varepsilon\underline{x}$ is given by

$$(D_\varepsilon\underline{x})(t) = \begin{cases} \underline{x}(t-\varepsilon) & \text{for } t > T_0+\varepsilon \\ \underline{0} & \text{for } T_0+\varepsilon \geqq t \geqq T_0 \end{cases}$$

An operator F from L_{pe}^q to $L_{pe}^{q'}$ is said to be *causal* if $P_TFP_T = P_TF$. The F is said to be *locally Lipschitz continouos* if

$$\sup_{P_T\underline{x}\neq P_T\underline{x}'} \frac{||P_T(F\underline{x}-F\underline{x}')||}{||P_T(\underline{x}-\underline{x}')||} < \gamma < \infty \qquad \text{for any } T\varepsilon S$$

Sec. 2.2. System Description and Well-Posedness

Let us consider the situation where we construct a mathematical model of a large-scale system by combining models of smaller parts. A general class of such models would be given in the form

$$\underline{e}_j = \underline{u}_j + \sum_{i=1}^{m} H_{ji}\underline{y}_i \qquad j=1, \cdots, n \qquad (2\text{-}1)$$

$$\underline{y}_i = \sum_{j=1}^{n} G_{ij}\underline{e}_j \qquad i=1, \cdots, m \qquad (2\text{-}2)$$

where $\underline{e}_j$, $\underline{u}_j \varepsilon L_{pe}^{r_j}$ and $\underline{y}_j \varepsilon L_{pe}^{q_i}$, G_{ij} is a locally Lipschitz continuous causal operator from $L_{pe}^{r_j}$ to $L_{pe}^{q_j}$ which satisfies $G_{ij}\underline{0} = \underline{0}$, and H_{ji} a locally Lipschitz continuous causal operator from $L_{pe}^{q_i}$ to $L_{pe}^{r_j}$ which satisfies $H_{ji}\underline{0} = \underline{0}$.

In the above, each G_{ij} and H_{ji} corresponds to the model of each part of the system (i.e., subsystem), and eqs. (2-1) and (2-2) to the physical connections among those subsystems where $\underline{u}_j$ are the external inputs to this interconnected system. Here, define R^r-valued functions $\underline{e}$, $\underline{u}$, and R^q-valued functions $\underline{y}$, respectively, by $\underline{e}(t) = (\underline{e}_1^T(t), \cdots, \underline{e}_n^T(t))^T$, $\underline{u}(t) = (\underline{u}_1^T(t), \cdots, \underline{u}_n^T(t))^T$, and $\underline{y}(t) = (\underline{y}_1^T(t), \cdots, \underline{y}_m^T(t))^T$ where $r = r_1+\cdots+r_n$, $q = q_1+\cdots+q_m$. Let operators G from L_{pe}^r to L_{pe}^q and H from L_{pe}^q to L_{pe}^r be defined by $G = (G_{ij})$ and $H = (H_{ji})$. Then, eqs. (2-1) and (2-2) can be simply written as

$$\underline{e} = \underline{u} + H\underline{y} \qquad (2\text{-}1')$$

$$\underline{y} = G\underline{e} \qquad (2\text{-}2')$$

In the following, we refer to the system given by eqs. (2-1) and (2-2) (equivalently by (2-1') and (2-2')) as the *CSF* (*C*omposite *s*ystem described by the *f*unctional relation), and, to the $\underline{e}_j$ as the j-th subvector of $\underline{e}$. The norm of $\underline{e}\varepsilon L_{pe}^r$ can be given by the norms of its subvectors as

$$||\underline{e}|| = \{ ||\underline{e}_1||^p + \cdots + ||\underline{e}_n||^p \}^{1/p} \qquad (2\text{-}3)$$

In addition to the above norm, we need to consider a weighted mean square norm

$$||\underline{e}||_W = \{ w_1||\underline{e}_1||^2 + \cdots + w_n||\underline{e}_n||^2 \}^{1/2} \qquad (2\text{-}4)$$

Note that the above two norms are equivalent. Also note that (2-1') and (2-2') are equivalint to, by the definition of equality in L^r_{pe} and by causality of G and H,

$$P_\tau \underline{e} = P_\tau \underline{u} + P_\tau H P_\tau \underline{y} \qquad \text{for any } \tau \in S \qquad (2\text{-}5)$$

$$P_\tau \underline{y} = P_\tau G P_\tau \underline{e} \qquad \text{for any } \tau \in S \qquad (2\text{-}6)$$

The above equations (2-5) and (2-6) are refered to as the *truncated system equations*.

Now, to characterize the suitability of the equations as a model of a physical system, the well-posedness is defined as follows.

Definition 2-1

The CSF is said to be *well-posed* if it possesses the properties (i), (ii) and (iii).

(i) There exists a unique pair of causal, locally Lipschitz continuous operators E from L^r_{pe} to L^r_{pe} and Y from L^r_{pe} to L^q_{pe} such that the pair $(\underline{e}, \underline{y})$ given by $\underline{e} = E\underline{u}$ and $\underline{y} = Y\underline{u}$ is the solution of CSF; i.e., these $\underline{e}$ and $\underline{y}$ satisfy eqs. (2-1') and (2-2').

(ii) Let CSF_ε be the system described by the equations obtained from (2-1) and (2-2) by replacing G_{ij} and H_{ji} with $D_{\varepsilon_{ij}} G_{ij}$ and $D_{\varepsilon'_{ji}} H_{ji}$, respectively. Then CSF_ε satisfies the above condition (i).

(iii) Let $(\underline{e}^\varepsilon, \underline{y}^\varepsilon)$ be the solution of CSF_ε and let

$$\varepsilon_0 = \max_{i=1,\cdots,n;j=1,\cdots,m} \{ \varepsilon_{ij}, \varepsilon'_{ij} \}. \text{ Then,}$$

$$\lim_{\varepsilon_0 \to 0} P_\tau \underline{e}^\varepsilon = P_\tau \underline{e}, \quad \lim_{\varepsilon_0 \to 0} P_\tau \underline{y}^\varepsilon = p_\tau \underline{y} \text{ for any } \tau \in S \qquad (2\text{-}7)$$

where $(\underline{e}, \underline{y})$ is the solution of CSF.

Condition (i) would be natural as a model of a physical system. Condition (ii) reflects the consideration that it takes a finite time, though it might be very small, for a signal to travel from one subsystem to another, and condition (iii) requires that the behavior of the solution does not change incontinuously even if we take those delay times into consideration. Comparison of our definition to the Vidyasagar's (1980) and Willems's (1971) will be done in Sec. 2.6.

Sec. 2.3. Uniform Instantaneous Gain

Willems (1971) used the uniform instantaneous gain of operators to guarantee well-posedness of single-loop systems. We follow the Willems's idea and adopt the following definition.

Definition 2-2

Let F be a locally Lipschitz continuous causal operator from L_{pe}^{q} and $L_{pe}^{q'}$. Let $T \in S$ and $\Delta T > 0$. Then, there exist two nonnegative numbers $M_{T,\Delta T}$ and $K_{T,\Delta T}$ such that

$$||(P_{T+\Delta T} - P_{T-\Delta T})(F\underline{x} - F\underline{x}')|| \leq M_{T,\Delta T}||(P_{T+\Delta T} - P_{T-\Delta T})(\underline{x} - \underline{x}')|| + K_{T,\Delta T}||P_{T-\Delta T}(\underline{x} - \underline{x}')|| \qquad (2\text{-}8)$$

for all $\underline{x}$, $\underline{x}' \in L_{pe}^{q}$. We refer to the infimum $M_{T,\Delta T}^{*}$ of the $M_{T,\Delta T}$ as the *uniform gain* of F in the interval $[T-\Delta T, T+\Delta T]$, and $\lim_{\Delta T \to +0} M_{T,\Delta T}^{*}$ as the *uniform instantaneous gain* of F at T.

The existence of $\lim_{\Delta T \to +0} M_{T,\Delta T}^{*}$ in the above definition is guaranteed by the next lemma.

Lemma 2-1

$$M_{T,\Delta T}^{*} \geq M_{T,\Delta T'}^{*}, \quad \text{if } \Delta T \geq \Delta T'.$$

[Proof] By causality

$$(P_{T+\Delta T'} - P_{T-\Delta T'})(F\underline{x} - F\underline{x}') = (P_{T+\Delta T'} - P_{T-\Delta T'})(FP_{T+\Delta T'}\underline{x} - FP_{T+\Delta T'}\underline{x}')$$

Therefore

$$\begin{aligned} ||(P_{T+\Delta T'} - P_{T-\Delta T'})(F\underline{x} - F\underline{x}')|| &\leq ||(P_{T+\Delta T} - P_{T-\Delta T})(FP_{T+\Delta T'}\underline{x} - FP_{T+\Delta T'}\underline{x}')|| \\ &\leq M_{T,\Delta T}||(P_{T+\Delta T} - P_{T-\Delta T})(P_{T+\Delta T'}\underline{x} - P_{T+\Delta T'}\underline{x}')|| \\ &\qquad + K_{T,\Delta T}||P_{T-\Delta T}(\underline{x} - \underline{x}')|| \\ &\leq M_{T,\Delta T}||(P_{T+\Delta T'} - P_{T-\Delta T'})(\underline{x} - \underline{x}')|| \\ &\qquad + (M_{T,\Delta T} + K_{T,\Delta T})||P_{T-\Delta T'}(\underline{x} - \underline{x}')|| \end{aligned}$$

From this, the conclusion of the lemma is evident. [Q.E.D.]

The uniform gain $M_{T,\Delta T}^{*}$ can be interpreted as giving, roughly speaking, the extent of the contribution of the difference between $\underline{x}$ and $\underline{x}'$ in the interval $[T-\Delta T, T+\Delta T]$ to the difference of the output in the same interval. So, the uniform instantaneous gain can be interpreted as the absolute value of the " feed-through " of the operator F. This interpretation is justified by the next theorems which give upper bounds of the uniform instantaneous gains of two important classes of operators.

Theorem 2-1 (Uniform instantaneous gain of linear systems)

Let the operator $F;\ L^{q}_{pe} \to L^{q'}_{pe}$ be defined by

$$(F\underline{x})(t) = \int_0^t \tilde{F}(t, \tau)\underline{x}(\tau)d\tau + F_0\underline{x}(t) + \sum_{k=1}^{\infty} F_k\underline{x}(t-\varepsilon_k) \qquad (2\text{-}9)$$

where $\tilde{F}(t, \tau)$ is an qxq' matrix whose elements are differentiable in both arguments and which are absolutely integrable in τ for any t, ε_k (k=1, 2, 3,$\cdots$) are positive numbers, and the constant matrices F_k (k=0, 1, 2, $\cdots$) are assumed to satisfy $\sum_{k=1}^{\infty} ||F_k|| < \infty$. Then, the uniform instantaneous gain of F is less than or equal to $||F_0||$.

Theorem 2-2 (Memoryless system)

Let the operator $F;\ L^{q}_{pe} \to L^{q'}_{pe}$ be defined by

$$(F\underline{x})(t) = \underline{\phi}(\underline{x}(t), t) \qquad (2\text{-}10)$$

where the function $\underline{\phi}(\ ,\ )$ satisfies

$$||\underline{\phi}(\underline{x}(t), t) - \underline{\phi}(\underline{x}'(t), t)|| \leq \beta(t)||\underline{x}(t) - \underline{x}'(t)|| \qquad (2\text{-}11)$$

Here, $\beta(t)$ is assumed to be piecewise continuous and bounded in any finite interval $[T_0, T]$. Then, the uniform instantaneous gain at T is less than or equal to

$$\beta^*(T) = \lim_{\Delta T \to 0} \sup_{t\varepsilon(T-\Delta T,\ T+\Delta T)} \beta(t) \qquad (2\text{-}12)$$

In Theorem 2-2, it should be noted that $\beta^*(T) = \beta(T)$ if $\beta(t)$ is continuous at T.

[Proof of Theorem 2-1] Choose ΔT so that $0< \Delta T < \frac{1}{2}\min_k \varepsilon_k$. Define the function $\underline{z}$ by

$$\underline{z}(t) = [(P_{T+\Delta T} - P_{T-\Delta T})F\underline{x}](t)$$
$$= \begin{cases} \int_0^{T+\Delta T} \tilde{F}(t, \tau)\underline{x}(\tau)d\tau + F_0\underline{x}(t) & \text{for } T-\Delta T \leq t \leq T+\Delta T \\ \underline{0} & \text{for } t < T-\Delta T \text{ or } t > T+\Delta T \end{cases}$$

where we employ the convention that $\tilde{F}(t, \tau)=0$ for $t<\tau$. Then,

$$||\underline{z}|| = (\int_{T-\Delta T}^{T+\Delta T} ||\int_0^{T+\Delta T} \tilde{F}(t, \tau)\underline{x}(\tau)d\tau + F_0\underline{x}(t)||^p dt\ )^{1/p}$$
$$\leq \{\int_{T-\Delta T}^{T+\Delta T} [||\int_0^{T-\Delta T} \tilde{F}(t, \tau)\underline{x}(\tau)d\tau|| + ||\int_{T-\Delta T}^{T+\Delta T} \tilde{F}(t, \tau)\underline{x}(\tau)d\tau||$$
$$+ ||F_0\underline{x}(t)||]^p dt\}^{1/p}$$

Using Minkowski's inequality

$$||\underline{z}|| \leq (\int_{T-\Delta T}^{T+\Delta T} ||\int_0^{T-\Delta T} \tilde{F}(t, \tau)\underline{x}(\tau)d\tau||^p dt)^{1/p}$$
$$+ (\int_{T-\Delta T}^{T+\Delta T} ||\int_{T-\Delta T}^{T+\Delta T} \tilde{F}(t, \tau)\underline{x}(\tau)d\tau||^p dt)^{1/p}$$
$$+ (\int_{T-\Delta T}^{T+\Delta T} ||F_0\underline{x}(t)||^p dt)^{1/p}$$

Using the result of Vidyasagar (1980, Lemma A_1) to estimate the second term.

$$||\underline{z}|| \leq K_{T,\Delta T}||P_{T-\Delta T}\underline{x}|| + c_{1T,\Delta T}^{1/p}\, c_{2T,\Delta T}^{1/p}\, ||(P_{T+\Delta T} - P_{T-\Delta T})\underline{x}||$$
$$+ ||F_0||\, ||(P_{T+\Delta T} - P_{T-\Delta T})\underline{x}||$$

where

$$c_{1T,\Delta T} = \sup_{\tau\varepsilon[T-\Delta T,\ T+\Delta T]} \int_{T-\Delta T}^{T+\Delta T} ||\tilde{F}(t, \tau)||dt$$
$$c_{2T,\Delta T} = \sup_{t\varepsilon[T-\Delta T,\ T+\Delta T]} \int_{T-\Delta T}^{T+\Delta T} ||\tilde{F}(t, \tau)||d\tau$$
$$\frac{1}{p} + \frac{1}{q} = 1$$

As $\tilde{F}(t, \tau)$ is a continuous function of t and τ, $c_{1T,\Delta T}$, $c_{2T,\Delta T} \to 0$ when $\Delta T \to 0$. Therefore, the uniform instantaneous gain of F is less than or equal to $||F_0||$ (Note that F is linear). [Q.E.D.]

[Proof of Theorem 2-2] By (2-11) and by the definition of $\beta^*(T)$, we have

$$||\underline{\phi}(\underline{\xi}, t) - \underline{\phi}(\underline{\xi}', t)|| \leq \{\beta^*(T) + \delta\}||\underline{\xi} - \underline{\xi}'||$$
$$t \varepsilon [T-\Delta T, T+\Delta T]$$

for small ΔT where $\delta \to 0$ when $\Delta T \to 0$. From this, the conclusion is evident.

[Q.E.D.]

Sec. 2.4. Sufficient Condition of Well-Posedness

Concerning the well-posedness of the CSF, we have the next theorem.

Theorem 2-3

Let the uniform instantaneous gains of G_{ij} and H_{ji} at $T\varepsilon S$ be $a_{ij}(T)$ and $b_{ji}(T)$, respectively. Define the *gain-product matrix* $\Theta(T) = (\theta_{jj'}(T))$ by

$$\theta_{jj'}(T) = \sum_{i=1}^{m} b_{ji}(T)a_{ij'}(T) \qquad j, j' = 1, \cdots, n \qquad (2\text{-}13)$$

Then, if the matrix $I - \Theta(T)$ is an M-matrix for all T, the CSF is well-posed in the sense of Definition 2-1.

[Proof] (i) Define $\overline{\Theta}(T, \delta) = (\overline{\theta}_{jj'}(T|\delta))$ for $\delta > 0$ by

$$\bar{\theta}_{jj'}(T|\delta) = \sum_{i=1}^{m} (b_{ji}(T) + \delta)(a_{ij'}(T) + \delta) \qquad (2\text{-}14)$$

Since $I - \bar{\Theta}(T|0) = I - \Theta(T)$ is an M-matrix, and since the elements of $\bar{\Theta}(T|\delta)$ continuously depend on δ, there is a positive number δ_0 such that $I - \bar{\Theta}(T|\delta)$ remains an M-matrix for all $\delta<\delta_0$. By the definition of the uniform instantaneous gain and by Lemma 2-1, the next inequalities hold true for sufficiently small ΔT and for some $K_{ij} \geqq 0$ and $K'_{ji} \geqq 0$:

$$||(P_{T+\Delta T} - P_{T-\Delta T})(G_{ij}\underline{x} - G_{ij}\underline{x}')||$$
$$\leqq (a_{ij}(T) + \delta)||(P_{T+\Delta T} - P_{T-\Delta T})(\underline{x} - \underline{x}')|| + K_{ij}||P_{T-\Delta T}(\underline{x} - \underline{x}')||$$
$$i=1, \cdots, n;\ j=1, \cdots, m: \quad \underline{x} \in L_{pe}^{r_j}\ ;\ \underline{x}' \in L_{pe}^{r_j} \qquad (2\text{-}15a)$$

$$||(P_{T+\Delta T} - P_{T-\Delta T})(H_{ji}\underline{y} - H_{ji}\underline{y}')||$$
$$\leqq (b_{ji}(T) + \delta)||(P_{T+\Delta T} - P_{T-\Delta T})(\underline{y} - \underline{y}')|| + K'_{ji}||P_{T-\Delta T}(\underline{y} - \underline{y}')||$$
$$i=i, \cdots, n;\ j=1, \cdots, m: \quad \underline{y} \in L_{pe}^{q_i};\ \underline{y}' \in L_{pe}^{q_i} \qquad (2\text{-}15b)$$

Now, let us assume that we have a unique solution $(\tilde{\underline{e}}, \tilde{\underline{y}})$ for the interval $[T_0, T']$ † where $T-\Delta T < T' < T+\Delta T$, and show that the solution can be extended up to $T+\Delta T$ uniquely. Substituting (2-6) into (2-5) and setting $\tau = T+\Delta T$,

$$P_{T+\Delta T}\underline{e} = P_{T+\Delta T}HGP_{T+\Delta T}\underline{e} + P_{T+\Delta T}\underline{u} \qquad (2\text{-}16)$$

By assumption, we have

$$P_{T'}\tilde{\underline{e}} = P_{T'}HGP_{T'}\tilde{\underline{e}} + P_{T'}\tilde{\underline{u}} \qquad (2\text{-}17)$$

Subtracting (2-17) from (2-16) and considering the facts that $\underline{e}$ is an extention of $\tilde{\underline{e}}$ (i.e. $P_{T'}\underline{e} = P_{T'}\tilde{\underline{e}}$) and that HG is causal, we obtain the equation

$$P_{T+\Delta T}\underline{e} = (P_{T+\Delta T} - P_{T'})HGP_{T+\Delta T}\underline{e} + P_{T'}\underline{e} + (P_{T+\Delta T} - P_{T'})\underline{u} \qquad (2\text{-}18)$$

Now, if $\underline{x}$ and $\underline{x}' \in L_{2e}^{r}$ satisfy

$$P_{T'}\underline{x} = P_{T'}\underline{x}' \qquad (2\text{-}19)$$

we obtain from (2-15a) and (2-15b) ††

---

†) To be exact, $\tilde{\underline{e}}(t)=\underline{0}$ and $\tilde{\underline{y}}(t)=\underline{0}$ for $t>T'$, and the pair $(P_{T'}\tilde{\underline{e}}, P_{T'}\tilde{\underline{y}})$ is the unique solution of the truncated equations (2-5) and (2-6) with $\tau=T'$.

††) Here, $T-\Delta T$ is replaced by T'. The validity of such replacement can be proved in parallel to the proof of Lemma 2-1.

$$||(P_{T+\Delta T} - P_{T-\Delta T})(HG\underline{x} - HG\underline{x}')_j||$$
$$\leq \sum_{i=1}^{m} (b_{ji}(T) + \delta) \sum_{k=1}^{n} \{(a_{ik}(T) + \delta)||(P_{T+\Delta T} - P_{T'})(\underline{x}_k - \underline{x}_k')||\}$$
$$j=1, \cdots, n \qquad (2\text{-}20)$$

where $(HG\underline{x} - HG\underline{x}')_j$ is the j-th subvector of $HG\underline{x} - HG\underline{x}'$. From (2-20) and (2-14), we obtain

$$||(P_{T+\Delta T} - P_{T'})(HG\underline{x} - HG\underline{x}')||_W$$
$$\leq \{\sum_{j=1}^{n} w_j [\sum_{k=1}^{n} \overline{\theta}_{jk}(T|\delta)||(P_{T+\Delta T} - P_{T'})(\underline{x}_k - \underline{x}_k')||\ ]^2\}^{1/2} \qquad (2\text{-}21)$$

where $||\ ||_W$ on the left-hand side is the weighted mean square norm defined by (2-4). If we set

$$\underline{\xi}^T = (||(P_{T+\Delta T} - P_{T'})(\underline{x}_1 - \underline{x}_1')||, \cdots, ||(P_{T+\Delta T} - P_{T'})(\underline{x}_n - \underline{x}_n')||)$$
$$(2\text{-}22)$$

the right-hand side of (2-21) can be expressed as $\underline{\xi}^T\overline{\Theta}^TW\overline{\Theta}\underline{\xi}$ where $W = \text{diag}(w_j)$; $j=1, \cdots, n$. Since $I - \overline{\Theta}(T|\delta)$ is an M-matrix, we can choose $W = \text{diag}(w_j)$ so that $W - \overline{\Theta}^TW\overline{\Theta}$ is positive definite (Lemma A-4). This means that there is a nonnegative constant $\gamma < 1$ such that

$$\underline{\xi}^T\overline{\Theta}^TW\overline{\Theta}\underline{\xi} \leq \gamma\underline{\xi}^TW\underline{\xi} \qquad (2\text{-}23)$$

As a result, we obtain

$$||(P_{T+\Delta T} - P_{T'})(HG\underline{x} - HG\underline{x}')||_W \leq \gamma||(P_{T+\Delta T} - P_{T'})(\underline{x} - \underline{x}')||_W \qquad (2\text{-}24)$$

for $\underline{x}$ and $\underline{x}'$ which satisfy (2-20). Here, let us consider a sequence $P_{T+\Delta T}\underline{e}_k$ $(k=1,2,\cdots)$ which is given by

$$P_{T+\Delta T}\underline{e}_{k+1} = (P_{T+\Delta T} - P_{T'})HGP_{T+\Delta T}\underline{e}_k + P_{T'}\tilde{\underline{e}} + (P_{T+\Delta T} - P_{T'})\underline{u} \quad (2\text{-}25)$$

From (2-24), $(P_{T+\Delta T} - P_{T'})HG$ is a contraction mapping. Therefore, $P_{T+\Delta T}\underline{e}_k$ converges to a limit $P_{T+\Delta T}\underline{e}$ which is the unique solution of (2-18). This solution clearly satisfies $P_{T'}\underline{e} = \tilde{\underline{e}}$. Thus, we have shown that $\underline{e}$ can be uniquely extended up to $T+\Delta T$.

Here, let $I_T=(T-\Delta T,\ T+\Delta T)$ be the open interval such that (2-15a) and (2-15b) hold true and $I-\overline{\Theta}(T|\delta)$ is an M-matrix. By the Heine-Borel finite-covering theorem, a finite closed interval $[\tilde{T}, \tilde{\tilde{T}}]$ can be covered by a finite number of intervals I_T. Therefore, the whole S can be covered by an infinite series of intervals I_{T_k} $(k=0,1,2,\cdots;\ T_0<T_1<\cdots<T_k<T_{k+1}\cdots)$. For the first interval

$[T_0, \Delta T_0]$, we can show that there is a unique solution $P_{\Delta T_0}\underline{e}$ for the truncated equations (2-5) and (2-6) with $\tau=\Delta T_0$ in the same way as the proof of the existence of the extention. By extending this solution interval by interval as shown in the preceding paragraph, we can conclude that there is the unique solution $(\underline{e}, \underline{y})$ of (2-1) and (2-2).

The causality of the operators E: $\underline{u} \mapsto \underline{e}$ and Y: $\underline{u} \mapsto \underline{y}$ is evident from the manner of constructing the solution $(\underline{e}, \underline{y})$. To show local Lipschitz continuity consider two inputs $\underline{u}$ and $\underline{u}'$, and let $\underline{e}$ and $\underline{e}'$ be the solutions corresponding to the inputs $\underline{u}$ and $\underline{u}'$, respectively. Then, by using (2-15a), (2-15b) and the assumption that $I-\Theta(T)$ is an M-matrix, $||(P_{T+\Delta T} - P_{T-\Delta T})(\underline{e} - \underline{e}')||$ can be bounded as

$$||(P_{T+\Delta T} - P_{T-\Delta T})(\underline{e} - \underline{e}')|| \leq a_T||P_{T-\Delta T}(\underline{e} - \underline{e}')|| + b_T||(P_{T+\Delta T} - P_{T-\Delta T})(\underline{u} - \underline{u}')|| \quad (2\text{-}26)$$

where a_T and b_T are some finite numbers. By using (2-26) repeatedly, we can show local Lipschitz continuity.

(ii) For the delayed operator $D_{\varepsilon_{ij}}G_{ij}$, we have

$$\begin{aligned} ||(P_{T+\Delta T} &- P_{T-\Delta T})(D_{\varepsilon_{ij}}G_{ij}\underline{x} - D_{\varepsilon_{ij}}G_{ij}\underline{x}')|| \\ &= ||(P_{T+\Delta T-\varepsilon_{ij}} - P_{T-\Delta T-\varepsilon_{ij}})(G_{ij}\underline{x} - G_{ij}\underline{x}')|| \\ &\leq ||(P_{T+\Delta T} - P_{T-\Delta T})(G_{ij}\underline{x} - G_{ij}\underline{x}')|| \\ &\qquad + ||(P_{T-\Delta T} - P_{T-\Delta T-\varepsilon_{ij}})(G_{ij}\underline{x} - G_{ij}\underline{x}')|| \\ &\leq ||(P_{T+\Delta T} - P_{T-\Delta T})(G_{ij}\underline{x} - G_{ij}\underline{x}')|| + \gamma_{T-\Delta T}||P_{T-\Delta T}(\underline{x} - \underline{x}')|| \quad (2\text{-}27) \end{aligned}$$

where, in deriving the last inequality, we used the local Lipshitz continuity of G_{ij}. From this, we can conclude that the uniform gain of $D_{\varepsilon_{ij}}G_{ij}$ in $[T-\Delta T, T+\Delta T]$ is less than or equal to that of G_{ij} in the same interval. For $D_{\varepsilon'_{ji}}H_{ji}$, we can obtain the same conclusion. So, just as in the derivation of (i), we can show that the solution $(\underline{\tilde{e}}^{\varepsilon}, \underline{\tilde{y}}^{\varepsilon})$ of the CS_{ε} on $[T_0, T']$ can be extended up to $T+\Delta T$. Namely, define the sequence $P_{T+\Delta T}\underline{e}^{\varepsilon}$ $(k=1,2,\cdots)$ by

$$P_{T+\Delta T}\underline{e}^{\varepsilon}_{k+1} = (P_{T+\Delta T} - P_{T'})H^{\varepsilon}G^{\varepsilon}\underline{e}^{\varepsilon}_k + P_{T'}\underline{\tilde{e}}^{\varepsilon} + (P_{T+\Delta T} - P_{T'})\underline{u} \quad (2\text{-}28)$$

where $G^{\varepsilon} = (D_{\varepsilon_{ij}} G_{ij})$ and $H^{\varepsilon} = (D_{\varepsilon'_{ji}} H_{ji})$.

Then, $P_{T+\Delta T}\underline{e}_k^{\varepsilon}$ converges to $P_{T+\Delta T}\underline{e}^{\varepsilon}$ which is the unique solution of

$$P_{T+\Delta T}\underline{e}^{\varepsilon} = (P_{T+\Delta T} - P_{T'})H^{\varepsilon}G^{\varepsilon}\underline{e}^{\varepsilon} + P_{T'}\tilde{\underline{e}}^{\varepsilon} + (P_{T+\Delta T} - P_{T'})\underline{u} \qquad (2\text{-}29)$$

From this, we can conclude that CS_{ε} also has Property (i).

(iii) By the fact that the uniform gains of $D_{\varepsilon_{ij}} G_{ij}$ and $D_{\varepsilon'_{ji}} H_{ji}$ are not greater than those of G_{ij} and H_{ji}, respectively, we can conclude that the sequence $P_{T+\Delta T}\underline{e}_k^{\varepsilon}$ converges uniformly with respect to ε_{ij} and ε'_{ji}. So, the order of $\lim_{\varepsilon_0 \to 0}$ and $k \to \infty$ can be interchanged. Therefore, if $\lim_{\varepsilon_0 \to 0} P_{T'}\tilde{\underline{e}}^{\varepsilon} = P_{T'}\underline{e}$, we obtain $\lim_{\varepsilon_0 \to 0} P_{T+\Delta T}\underline{e}^{\varepsilon} = P_{T+\Delta T}\underline{e}$. By induction, we can conclude that $\lim_{\varepsilon_0 \to 0} P_{\tau}\underline{e} = P_{\tau}\underline{e}$ for any $\tau \varepsilon S$. [Q.E.D.]

Sec. 2.5. Graph-Theoretic Expression of the Well-Posedness Condition and Relation to the Vidyasagar's Result

Here, let us study the graph-theoretic implication of the well-posedness condition obtained in the preceding section. The next theorem tells that the M-matrix condition of Theorem 2-3 can be tested in two steps: i.e. a preliminary study of the digraph obtained from the system equation and M-matrix tests of matrices with smaller sizes.

Theorem 2-4

Let *G(CSF)* be the digraph with n+m nodes labelled as $e_1, \cdots, e_n; y_1, \cdots, y_m$ and with 2nm edges connecting e_j to y_i and y_i to e_j ($i=1, \cdots, m; j=1, \cdots, n$). Let $G_0(CSF)$ be digraph obtained from *G(CSF)* by striking out edges as follows.

(a) If the uniform instantaneous gain $a_{ij}(T)$ of G_{ij} is zero for all $T \varepsilon S$, strike out the edge from e_j to y_i.

(b) If the uniform instantaneous gain $b_{ji}(T)$ of H_{ji} is zero for all $T \varepsilon S$, strike out the edge from y_i to e_j. Let $G_k(CSF)$ ($k=1, \cdots, s$) be the strongly connected parts of $G_0(CSF)$. Associated with $G_k(CSF)$, consider the smaller size composite system CSF_k given by

$$\underline{e}_j = \underline{u}_j + \sum_{i\varepsilon I_k} H_{ji}\underline{y}_i \qquad j\varepsilon J_k \tag{2-30}$$

$$\underline{y}_i = \sum_{j\varepsilon J_k} G_{ij}\underline{e}_j \qquad i\varepsilon I_k \tag{2-31}$$

where J_k is the set of j such that the node e_j belongs to $G_k(CSF)$, and I_k is the set of i such that the node y_i belongs to $G_k(CSF)$. Let $\Theta_k(T)$ be the loop-gain-product matrix of CSF_k. If $I - \Theta_k(T)$ is an M-matrix for all $T\varepsilon S$ and $k=1, \cdots, s$, the CSF is well-posed in the sense of Definition 2-1.

[Proof] Since $G_0(CSF)$ has no edges connecting e_j to $e_{j'}$ nor y_j to $y_{j'}$, the strongly connected part G_k is uniquely characterized by the e_j's which belong to G_k. To find those e_j's, eliminate the nodes y_i from $G_0(CSF)$; i.e. connect e_j to $e_{j'}$ directly if there is at least one node y_i such that $G_0(CSF)$ has edges from e_j to y_i and y_i to $e_{j'}$. Then, we obtain a digraph $\overline{G}_0(CSF)$ with n nodes $e_1, \cdots, e_n$. If we consider the fact that $a_{ij}(T) \geqq 0$ and $b_{ji}(T) \geqq 0$ and the way of constructing $\overline{G}_0(CSF)$, it would be evident that the node-to-node incidence matrix of $\overline{G}_0(CSF)$ is obtained from the loop-gain-product matrix $\Theta(T)$ by replacing non-zero elements with 1.† Now, by renumbering e_j's, we can bring the non-negative matrix $\Theta(T)$ to the normal form which is lower block triangular given in Fig. 2-1 (Seneta 1973). As shown in Seneta (1973), each strongly connected subgraph of $\overline{G}_0(CSF)$ corresponds to a diagonal block Θ_k of this normal form. Now, $I - \Theta(T)$ is an M-matrix if and only if its principal minors are all positive. Therefore, from the form of $\Theta(T)$ given in Fig. 2-1, $I - \Theta(T)$ is an M-matrix if and only if $I - \Theta_k(T)$ is an M-matrix for $k=1, \cdots, s$. Since $\Theta_k(T)$ is nothing but the loop-gain-product matrix of the CSF_k, we can obtain the conclusion of the theorem. [Q.E.D.]

$$\begin{bmatrix} \Theta_1 & 0 & \cdots & 0 \\ x & \Theta_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ x & \cdots & x & \Theta_s \end{bmatrix}$$

Fig. 2-1 The normal form of $\Theta(T)$
The x indicates a block which is not necessarily zero, and Θ_k is a square block.

The next corollary is evident.

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†) By the term " non-zero element ", we mean the $\theta_{jj'}(T)$ which is not zero for some T.

Corollary 2-1

If $G_0(CSF)$ has no cycles, the CSF is well-posed in the sense of Definition 2-1.

Now, let us compare the above result with that of Vidyasagar. He defined smoothing operator as follows.

Definition 2-3

A causal, locally Lipschitz continuous operator F from L^q_{pe} to $L^{q'}_{pe}$ is said to be smoothing if, for any $\tau > T_0$ and $\alpha > 0$, there exists δ such that $0 < \delta < \tau$ and

$$\sup_{P_{t+\delta}\underline{x} \neq P_{t+\delta}\underline{x}'} \frac{\| P_{t+\delta}(FP_{t+\delta} - FP_t)\underline{x} - P_{t+\delta}(FP_{t+\delta} - FP_t)\underline{x}' \|}{\| P_{t+\delta}\underline{x} - P_{t+\delta}\underline{x}' \|} < \alpha$$

$$\text{for any } t \in [0, \tau-\delta] \qquad (2\text{-}32)$$

Using this concept, he obtained the next result.

Theorem 2-5 (Vidyasagar 1980)†

Let $G_V(CSF)$ be the digraph obtained from $G(CSF)$ by striking out the edge from e_j to y_i if G_{ij} is smoothing and the edge from y_i to e_j if H_{ji} is smoothing. If $G_V(CSF)$ has no cycles,†† the CSF has the properties (i) and (ii) of Definition 2-1.

We can show that the above theorem is included in our Theorem 2-4. As preliminary, let us prove the next lemma.

---

†) He treated only the special case of the CSF in which m=n and $G_{ij}=0$ for $i \neq j$. It is evident that his result can be extended to general CSF. The fulfilment of the property (ii) is not included in the main body of his theorem, but is noted in the remarks after the proof of his lemma 4.

††) He required $G_V(CSF)$ has no self-loops either, but this requirement is automatically satisfied because of the special property of $G_V(CSF)$ mentioned in the proof of Theorem 2-4.

Lemma 2-2

If F is smoothing, its uniform instantaneous gain M_T^* is 0 for all $T \in S$.

[Proof] Choose $\tau > T$. From (2-32) we obtain

$$||(P_{t+\delta} - P_t)(FP_{t+\delta}\underline{x} - FP_{t+\delta}\underline{x}')|| - ||(P_{t+\delta} - P_t)(FP_t\underline{x} - FP_t\underline{x}')||$$
$$\leqq \alpha||(P_{t+\delta} - P_t)(\underline{x} - \underline{x}')|| + \alpha||P_t(\underline{x} - \underline{x}')||$$

By local Lipschitz continuity of F, we obtain

$$||(P_{t+\delta} - P_t)(F\underline{x} - F\underline{x}')|| \leqq \alpha||(P_{t+\delta} - P_t)(\underline{x} - \underline{x}')|| + (\alpha + K')||P_t(\underline{x} - \underline{x}')||$$

By setting $t = T - \frac{\delta}{2}$ and $\Delta T = \frac{\delta}{2} < \tau - T$ we obtain

$$||(P_{T+\Delta T} - P_{T-\Delta T})(F\underline{x} - F\underline{x}')|| \leqq \alpha||(P_{T+\Delta T} - P_{T-\Delta T})(\underline{x} - \underline{x}')||$$
$$+ K_T||P_{T-\Delta T}(\underline{x} - \underline{x}')||$$

Thus, we obtain that the uniform gain $M_{T,\Delta T}^*$ of F is not greater than α for all $T \in [T_0, \tau - \Delta T]$. Since τ can be chosen arbitrarily large and α can be made arbitrarily small by decreasing δ, we can conclude that $G_0(CSF)$ is a subgraph of $G_V(CSF)$. Therefore, if $G_V(CSF)$ has no cycles, $G_0(CSF)$ has neither. So, by Corollary 2-1, the CSF is well-posed in the sense of Definition 2-1, i.e. it possesses properties (i), (ii) and (iii), if the CSF satisfies the requirements of Theorem 2-5. Thus we have shown that Vidyasagar's result is included in Corollary 2-1. [Q.E.D.]

Sec. 2.6. Remarks

Our definition of well-posedness (Definition 2-1) is stronger than Vidyasagar's in requiring (ii) and (iii). We adopted this definition because of the following reason. One of the representative examples of ill-posed models can be obtained when we make a loop consisting of two adders in the analogue computer operation. If we regard an adder as executing multiplication by a constant number, the system should have a unique solution if the loop gain is not 1. However, the actual system will be saturated when the loop gain is greater than 1. This phenomenon can be explained by taking the parasitic delays into consideration (J.C. Willems 1971), i.e. we can easily find that the model does not possess the properties (ii) and (iii) in this case. Since the situation observed in this example is fairly plausible in other engineering models, we think (ii) and (iii) are necessary requirements for well-posedness.

Willems included continuity of solutions with respect to a small change of the operators as a requirement for well-posedness (J.C. Willems 1971). The reason why we did not include this condition is rather technical; i.e. it seems difficult to define abstractly the " small " changes of operators so that we can derive a pratical condition which is applicable to various situations.
Theorem 2-3 seems to remain true even if we include such a requirement in the definition of well-posedness.

If we set n=m=1, Theorem 2-3 says that the system

$$\underline{e} = \underline{u} + H\underline{y} \quad , \quad \underline{y} = G\underline{e}$$

is well-posed if 1 > ab where a and b are the uniform instantaneous gains of G and H, respectively. This is nothing but the case 1 of Theorem 4.1 of Willems (1971).

Smoothness of an operator F given in Definition 2-4 implies, roughly speaking, that the uniform instantaneous gain M_T^* of F is zero for all $T \varepsilon S$ and that the convergence of $M_{T,\Delta T}^*$ to M_T^* is uniform with respect to T in the finite interval $T \varepsilon [T_0, \tau]$.

Chapter 3 Composite System Method in the Stability Analysis of Nonlinear Systems

In this chapter, we obtain a stability condition of general composite systems and an estimate of their stability regions. The results obtained in this chapter is used to analyze the stability of specific systems in Chapters 5 and 6.

Sec. 3.1. Composite System Method

First, let us roughly explain the composite system method. For the detailed reviews, readers are referred to , for example, Šiljak (1978). The composite system method has been developed to analyze large-scale systems. As a large-scale system usually has many variables and is composed of a large number of parts (subsystems) which interact in various ways, we encounter the following problems when we try to analyze it in one piece. The computational efforts become enormous because of the high dimensionality and the analysis becomes difficult because of complexity. For example, it is difficult to get a clear view over the influence of various factors on the behavior of the whole system, which is important to synthesize control systems. To remove these difficulities, the composite system method has been developed. This method follows the next procedure.

(Step 1) Decompose a large-scale system into smaller subsystems and their interconnections.

(Step 2) Analyze each subsystem independently as if their interconnections do not exist, and

(Step 3) analyze their interconnection parts.

(Step 4) Combine the results of Step 2 and Step 3 to reduce the properties of the whole.

Next, let us study the composite system method in the stability analysis. As we have said in Introduction, in the Lyapunov stability analysis of composite systems, there are two methods; the vector Lyapunov function method and the scalar Lyapunov function method. As we use the scalar Lyapunov function method in this thesis, let us explain this method according to Araki (1978a).

Let us consider a dynamical system given by

$$\dot{\underline{x}} = \underline{f}(\underline{x},\ t) \qquad (3\text{-}1)$$

where $\underline{f}(\underline{x},\ t)$ satisfies

$$\underline{f}(\underline{0},\ t) = \underline{0} \qquad \text{for all } t \qquad (3\text{-}2)$$

From (3-2), $\underline{x} = \underline{0}$ is an equilibrium. We want to assure Lyapunov stability of this equilibrium. When the dimensionality of this system is high and $\underline{f}$ is a nonlinear time-varying function, it is usually impossible to construct a Lyapunov function suited to this system. Let us apply the composite system method. We will decompose (3-1) as follows.

$$\begin{aligned} \dot{\underline{x}}_1 &= \underline{\tilde{f}}_1(\underline{x}_1,\ t) + \underline{\tilde{g}}_1(\underline{x},\ t) \\ &\ \ \vdots \\ \dot{\underline{x}}_m &= \underline{\tilde{f}}_m(\underline{x}_m,\ t) + \underline{\tilde{g}}_m(\underline{x},\ t) \end{aligned} \qquad (3\text{-}3)$$

where $\underline{x}_j$ is an m_j-vector, $n_1 + n_2 + \cdots + n_m = m$ and

$$\underline{x} = (\underline{x}_1^T,\ \cdots,\ \underline{x}_m^T)$$

and $\underline{\tilde{f}}_j(\underline{x}_j,\ t)$ and $\underline{\tilde{g}}_j(\underline{x},\ t)$ satisfy

$$\underline{\tilde{f}}_j(\underline{0},\ t) = \underline{0},\quad \underline{\tilde{g}}_j(\underline{0},\ t) = \underline{0} \quad j=1,\ \cdots,\ m \quad \text{for all } t$$

This system can be viewed as the composition of subsystems

$$\dot{\underline{x}}_j = \underline{\tilde{f}}_j(\underline{x}_j,\ t) + \underline{y}_j \qquad j=1,\ \cdots,\ m \qquad (3\text{-}4)$$

which are interconnected by

$$\underline{y}_j = \underline{\tilde{g}}_j(\underline{x},\ t) \qquad j=1,\ \cdots,\ m \qquad (3\text{-}5)$$

We refer to this system (3-3) as the *CSD* (*C*omposite *s*ystem *d*escribed by *d*ifferential equations). It will prove convenient to consider a free dynamical system

$$\dot{\underline{x}}_j = \underline{\tilde{f}}_j(\underline{x}_j,\ t) \qquad (3\text{-}6)$$

We refer to this system (3-6) as the *j-th isolated subsystem.*

Now, let us make a general consideration about how to analyze the CSD. As the result of decomposition, we can expect that construction of Lyapunov functions for the isolated subsystems is comparatively easy. So, let us assume that a Lyapunov function $v_j(x_j,\ t)$ is obtained for every isolated subsystem and let us consider to use their weighted sum

$$v(\underline{x}, t) = d_1 v_1 + \cdots + d_m v_m \tag{3-7}$$
$$d_j > 0 , \quad j=1, \cdots, m$$

as a candidate for the Lyapunov function of the CSD. Since $v_j(\underline{x}_j, t)$ is positive-definite, decrescent and radially unbounded, $v(\underline{x}, t)$ satisfies such properties. Therefore, the only condition we have to examine is the negative-definiteness of $\dot{v}|_{(3-3)}$ (derivative along the solution of (3-3)). From (3-7), we obtain

$$\dot{v}|_{(3-3)} = \sum_{j=1}^{m} d_j \dot{v}_j|_{(3-3)} \tag{3-8}$$

where

$$\begin{aligned} \dot{v}_j|_{(3-3)} &= \frac{\partial v_j}{\partial t} + (\mathrm{grad}_j v_j)^T \{\underline{\tilde{f}}_j(\underline{x}_j, t) + \underline{\tilde{g}}_j(\underline{x}, t)\} \\ &= \dot{v}_j|_{(3-6)} + (\mathrm{grad}_j v_j)^T \underline{\tilde{g}}_j(\underline{x}, t) \end{aligned} \tag{3-9}$$

$$\mathrm{grad}_j v_j = \left(\frac{\partial v_j}{\partial x_1^{(j)}}, \cdots, \frac{\partial v_j}{\partial x_{m_j}^{(j)}}\right)^T \tag{3-10}$$

and $x_k^{(j)}$ is the k-th component of $\underline{x}_j$. Eqs. (3-8) and (3-9) suggest the next procedure of the stability analysis.

(Step 1; Analysis of subsystem) Obtain a bound for $\dot{v}_j|_{(3-6)}$.

(Step 2; Analysis of interconnection) Obtain a bound for $\underline{\tilde{g}}_j(\underline{x}_j, t)$.

(Step 3; Condition for interconnected structure) Test a condition for the existence of d_j's that assure negative-definiteness of (3-8) from the above bounds.

The next theorem which follows the above procedure is given by Araki (1978a). In the nect section, we will give a more general theorem.

Theorem 3-1 (Araki 1978a)

The origin of the CSD is uniformly asymptotically stable in the large if the next three conditions are satisfied.

(a) For each isolated subsystem, there is a positive-definite, decrescent, radially unbounded function $v_j(\underline{x}_j, t)$ with continuous partial derivatives such that

$$\dot{v}_j|_{(3-6)} \leqq -\alpha_j [u_j(\underline{x}_j)]^2 - u_{0j}(\underline{x}_j), \quad \underline{x}_j \in R^{m_j} \tag{3-11}$$

where α_j is a constant, $u_j(\underline{x}_j)$ a nonnegative-definite function and $u_{0j}(\underline{x}_j)$ a positive-definite function.

(b) There are constants β_{jk} satisfying $\beta_{jk} \geqq 0$ for $j \neq k$ such that

$$(\text{grad}_j v_j)^T \underline{\tilde{g}}_j(\underline{x}, t) \leqq u_j(\underline{x}_j) \sum_{k=1}^{m} \beta_{jk} u_k(\underline{x}_k) \tag{3-12}$$

(c) The mxm matrix $A = (a_{jk})$ given by

$$a_{jj} = \alpha_j - \beta_{jj} \; ; \quad a_{jk} = - \beta_{jk} \quad (j \neq k) \tag{3-13}$$

is an M-matrix.

Stability conditions obtained by the composite system method usually have the following defects.

(a) The stability condition tends to be more conservative if we decompose the system into larger numbers of subsystems.

(b) We have no practical means to decompose a large-scale system.

(c) We have no practical means to construct a suitable Lyapunov function for each subsystem.

(d) The result obtained by the composite system method is sometimes blamed to be too conservative for practical use.

In Introduction, we mentioned that generally the stability conditions which can be obtained by the vector Lyapunov function method with linear, time-invariant comparison systems are always reduced by the scalar Lyapunov function method (Araki 1978b). We will examine this fact in the concrete about Theorem 3-1 and Šiljak's condition (1972) where the latter condition is obtained using the vector Lyapunov function method. It should be noted that Šiljak's condition was used by Weissenberger to estimate the stability region of composite systems.

Šiljak assumed that, for each isolated subsystem (3-6), there exists a first order Lyapunov function $v_j'(\underline{x}_j, t)$ $(j=1, \cdots, m)$ which satisfies

$$\begin{aligned} &\eta_{j1}\|\underline{x}_j\| \leqq v_j' \leqq \eta_{j2}\|\underline{x}_j\| \\ &\dot{v}_j'|_{(3-6)} \leqq -\eta_{j3}\|\underline{x}_j\| \\ &\|\text{grad}_j v_j'\| \leqq \eta_{j4} \qquad \text{for } \underline{x}_j \in R^{m_j} \end{aligned} \tag{3-14}$$

and that the interconnection functions satisfy

$$||g_j(\underline{x}, t)|| \leq \sum_{k=1}^{m} \beta_{jk}||\underline{x}_k|| \qquad (3\text{-}15)$$

where $||\underline{x}||$ *denotes the Euclidean norm of* $\underline{x}$. *He derived that the CSD is asymptotically stable in the large if the matrix* $A' = (a'_{jk})$*;*

$$a'_{jj} = -\eta_{j2}^{-1}\eta_{j3} + \beta_{jj}\eta_{j1}^{-1}\eta_{j4};\ a'_{jk} = \beta_{jk}\eta_{j1}^{-1}\eta_{j4}\ (j \neq k)$$

satisfies the Hick's condition (namely, $-A'$ *is an M-matrix).*

Here, put $v_j = v_j'^2/(2\eta_{j2}\eta_{j4})$ and $u_j = ||\underline{x}_j||$. Then, our assumptions (3-11) are satisfied with $\alpha_j = -\eta_{j1}\eta_{j3}/(\eta_{j2}\eta_{j4})$. Since $-A' = \mathrm{diag}(\eta_{j4})^{-1}A\ \mathrm{diag}(\eta_{j1})$ (A is given by (3-13)), the matrix A is an M-matrix if A' satisfies the Hick's condition (Lemma A-3). This means that Araki's condition is always satisfied if the Šiljak's is satisfied. Next let us consider the inverse. Assume that (3-11) is satisfied with $||\mathrm{grad}_j v_j|| \leqq u_j$ and $u_j = ||\underline{x}_j||$. Put $v_j' = \sqrt{v_j}$. In many cases v_j is a second order function of $\underline{x}_j$, and so, there exist η_{j1} and η_{j2} such that $\eta_{j1}||\underline{x}_j|| \leqq v_j' \leqq \eta_{j2}||\underline{x}_j||$. Then Šiljak's assumptions are satisfied with $\eta_{j3} = \alpha_j/(2\eta_{j2})$ and $\eta_{j4} = 1/(2\eta_{j1})$. Thus we obtain

$$-A' = \mathrm{diag}(\eta_{j1}^{-2})[\delta_{jk}(\eta_{j1}/\eta_{j2})^2\alpha_j - \beta_{jk}]$$

where δ_{jk} is a kronecker delta. As $(\eta_{j1}/\eta_{j2})^2 \leqq 1$, A is more likely to be an M-matrix than $-A'$. This means that, with the above choice of v', the Šiljak's condition is not necessarily satisfied even if Araki's is satisfied.

Sec. 3.2. New Stability Criterion of Composite Systems

Theorem 3-2

The origin of the CSD is asymptotically stable if the next three conditions are satisfied.

(a) For each isolated subsystem, there is a decrescent, positive-definite function $v_j(\underline{x}_j, t)$ with continuous partial derivatives such that

$$\dot{v}_j(\underline{x}_j, t)|_{(3\text{-}6)} \leqq -\kappa_j[u_{1j}(\underline{x}_j)]^2 + 2\lambda_j u_{1j}(\underline{x}_j)u_{2j}(\underline{x}_j) - \mu_j[u_{2j}(\underline{x}_j)]^2 - u_{0j}(\underline{x}_j) \quad \underline{x}_j \in R_j \quad j=1, \cdots, m \qquad (3\text{-}16)$$

where κ_j and μ_j are positive constants, u_{0j} positive-definite and R_j is the set including the origin $\underline{x}_j = \underline{0}$ as an inner point.

(b) There are constants γ_{jk} such that

$$(\mathrm{grad}_j v_j)^T \underline{\tilde{g}}_j(\underline{x},\ t) \leq 2u_{1j}(\underline{x}_j) \sum_{k=1}^{m} \gamma_{jk} u_{2k}(\underline{x}_k) \tag{3-17}$$

$$\underline{x}_k \in R_k \quad k=1,\ \cdots,\ m$$

(c) The matrix $K^{1/2}M^{1/2} - \tilde{\Gamma}$ is an M-matrix where

$$K = \mathrm{diag}(\kappa_j),\ M = \mathrm{diag}(\mu_j) \tag{3-18}$$

$$\tilde{\Gamma} = (\tilde{\gamma}_{jk})\ ;\ \tilde{\gamma}_{jj} = |\gamma_{jj} + \lambda_j|\ ,\quad \tilde{\gamma}_{jk} = |\gamma_{jk}| \qquad (j \neq k)$$

$$j,\ k = 1,\ \cdots,\ m$$

If $R_j = R^{m_j}$, the origin of the CSD is asymptotically stable " in the large".

[Proof] Define $v(\underline{x},\ t)$ by (3-7). By (3-8), (3-9) and (3-19), we obtain

$$\dot{v}|_{(3\text{-}6)} \leq - \sum_{j=1}^{m} d_j\kappa_j u_{1j}^2 + 2 \sum_{j=1}^{m} d_j\lambda_j u_{1j}u_{2j} - \sum_{j=1}^{m} d_j\mu_j u_{2j}^2$$
$$+ 2 \sum_{j=1}^{m} \sum_{k=1}^{m} d_j\gamma_{jk}u_{1j}u_{2k} - \sum_{j=1}^{m} d_j u_{0j} \quad \text{for } \underline{x} \in R \tag{3-19}$$

where R is the direct product of $R_1,\ \cdots,\ R_m$. Here, let $\underline{u}$ be the 2m-vector given by

$$\underline{u} = (u_{11},\ \cdots,\ u_{1m},\ u_{21},\ \cdots,\ u_{2m})^T \tag{3-20}$$

Then, eq. (3-19) can be written as

$$\dot{v}|_{(3\text{-}6)} \leq - \underline{u}^T V \underline{u} - \sum_{j=1}^{m} d_j u_{0j} \quad \text{for } \underline{x} \in R \tag{3-21}$$

where V is the 2m x 2m matrix given by

$$V = \begin{pmatrix} DK & - D(\Lambda + \Gamma) \\ - (\Lambda + \Gamma)^T D & DM \end{pmatrix} \tag{3-22}$$

where

$$D = \mathrm{diag}(d_j) \quad j=1,\ \cdots,\ m$$

The above V can be expressed as

$$V = \begin{pmatrix} I & 0 \\ (\Lambda + \Gamma)^T K^{-1} & I \end{pmatrix}^{-1} \begin{pmatrix} DK & 0 \\ 0 & V_0 \end{pmatrix} \begin{pmatrix} I & K^{-1}(\Lambda + \Gamma) \\ 0 & I \end{pmatrix}^{-1} \tag{3-23}$$

where

$$V_0 = M^{1/2}DM^{1/2} - (\Lambda + \Gamma)^T K^{-1/2}DK^{-1/2}(\Lambda + \Gamma) \tag{3-24}$$

By assumption (c) and by the property of M-matrices given by Lemma A-4, we can choose D so that V_0 is positive-semi-definite. Then, the positive-semi-definiteness of V follows from (3-23). As u_{0j} is positive-definite, the right-hand side of (3-21) is negative-definite for $\underline{x}\varepsilon R$. Since $v(\underline{x}, t)$ is positive-definite and decrescent from the definition (3-7), and since R is a set including the origin $\underline{x} = \underline{0}$ as an inner point, the origin of the CSD is asymptotically stable. The last part of the theorem is evident. [Q.E.D.]

Let us show that Theorem 3-2 includes Araki's condition (Theorem 3-1) as a special case. Assume that the conditions of Theorem 3-1 are satisfied, we obtain from (3-11) and (3-12)

$$\dot{v}_j|_{(3-6)} \leqq - (\alpha_j - \beta_{jj})(\frac{u_j}{\sqrt{2}})^2 + 2\beta_{jj}(\frac{u_j}{\sqrt{2}})(\frac{u_j}{\sqrt{2}}) - (\alpha_j - \beta_{jj})(\frac{u_j}{\sqrt{2}})^2 - u_{0j} \tag{3-25}$$

$$(\text{grad}_j v_j)^T \underline{\tilde{g}}_j(\underline{x}, t) \leqq 2(\frac{u_j}{\sqrt{2}}) \sum_{k=1}^{m} \beta_{jk}(\frac{u_k}{\sqrt{2}}) \tag{3-26}$$

and from (3-13)

$$a_{jj} = \alpha_j - \beta_{jj} = \sqrt{\alpha_j - \beta_{jj}}\sqrt{\alpha_j - \beta_{jj}} - |\beta_{jj} - \beta_{jj}|$$

$$a_{jk} = \beta_{jk} \quad (j \neq k) \tag{3-27}$$

From the above equations (3-25)-(3-27), the conditions of Theorem 3-2 are satisfied with

$$u_{1j} = \frac{u_j}{\sqrt{2}}, \quad u_{2j} = \frac{u_j}{\sqrt{2}}$$

$$\kappa_j = \alpha_j - \beta_{jj}, \quad \lambda_j = \beta_{jj}, \quad \mu_j = \alpha_j - \beta_{jj} \tag{3-28}$$

$$\gamma_{jk} = \beta_{jk}$$

Therefore, Theorem 3-2 includes Theorem 3-1.

From the above result, Theorem 3-2 is potential to give a sharper result than Theorem 3-1. In Chapter 7, we will compare their sharpness quantitatively by applying these theorems to a numerical example (Example 2).

Sec. 3.3. Estimate of the Stability Region of Composite Systems

In this section, we consider the problem of obtaining an estimate of the stability region of the CSD. We attack this problem using the stability condition of Theorem 3-2. We derive an estimate of the stability region using the knowledge about subsystems. The application of Šiljak's condition and Theorem 3-1 to the estimation of the stability region has already been reported by Weissenberger (1973) and Saeki & Araki (1979), respectively.

3.3.1. Fundamental Theorem

Theorem 3-3

Let us define a subset S_j by

$$S_j = \{\underline{x}_j \mid v_j(\underline{x}_j) \leqq v_{0j}\} \tag{3-29}$$

and assume that S_j is contained in R_j. Assume that $K^{1/2}M^{1/2} - \tilde{\Gamma}$ given by (3-18) is an M-matrix and let $D = \mathrm{diag}(d_j)$ be a diagonal matrix with $d_j > 0$ such that $M^{1/2}DM^{1/2} - \tilde{\Gamma}^T K^{-1/2} D K^{-1/2} \tilde{\Gamma}$ is positive semi-definite. Then, the set S_x

$$S_x = \{\underline{x} \mid v(\underline{x}) \leqq \tilde{\pi}_0\} \tag{3-30}$$

is included in the stability region of $\underline{x} = \underline{0}$ of the CSD. Here, $v(\underline{x})$ is given by (3-7) and

$$\tilde{\pi}_0 = \min_{1 \leqq j \leqq m} d_j v_{0j} \tag{3-31}$$

[Proof] Since $v(\underline{x})$ is a positive-definite function and $\dot{v}|_{(3-6)}$ is negative-definite for d_j's satisfying the condition of the theorem, we can apply the result of Weissenberger (1973). Namely the region given by

$$S'_x = \{\underline{x} \mid v \leqq \tilde{\pi}'_0\} \tag{3-32}$$

is included in the stability region of $\underline{x} = \underline{0}$ where

$$\tilde{\pi}'_0 = \min_{\underline{v} \in U} v \tag{3-33}$$

$$U = \bigcup_{j=1}^{m} \{\underline{v} \mid v_j = v_{0j},\ v_k \leqq v_{0k},\quad k \neq j,\ k=1, \cdots, m\}$$

and $\underline{v} = (v_1, \cdots, v_m)^T$. From (3-7) we can easily derive that the value of (3-33) is equal to $\tilde{\pi}_0$ given by (3-31). Therefore we obtain the conclusion of the theorem. [Q.E.D.]

In the above theorem, it should be noted that there are many d_j's which make $M^{1/2}DM^{1/2} - \tilde{\Gamma}^T K^{-1/2} DK^{-1/2} \tilde{\Gamma}$ positive-semi-definite and that the estimate S_x given by (3-30) depends upon the values of d_j's. So, we are faced to the problem of finding d_j's which give the best estimate. We examine this problem in 3.3.2. It should be noted that S_x also depends upon S_j's given by (3-29) (which are the estimate of the subsystems' stability regions). But in order to obtain some concrete results about this sort of relation between the overall estimate and the local estimates, we need to fix the forms of the nonlinear functions $\underline{\tilde{f}}_j$ and $\underline{\tilde{g}}_j$. Such study must be done for each type of large-scale systems and is not included in the scope of this section.

3.3.2. Determination of d_j's

We first clarify what we mean by the " best estimate " of the stability region and then propose a method to determine d_j's in this direction.

In order to simplify the discussion, define the vector $\underline{z}$ by

$$z_j = v_j(\underline{x}_j)/v_{0j} \qquad j=1, \cdots, m \tag{3-34}$$

Then, by (3-7) and (3-30), we can easily derive that the set S_x of Theorem 3-3 is determined by the condition

$$\frac{\tilde{\pi}_1}{\tilde{\pi}_0} z_1 + \cdots + \frac{\tilde{\pi}_m}{\tilde{\pi}_0} z_m \leqq 1 \tag{3-35}$$

$$z_j \geqq 0 \quad \text{for } j=1, \cdots, m$$

where $\tilde{\pi}_j$ is the j-th element of the vector $\underline{\tilde{\pi}}$ and given by

$$\tilde{\pi}_j = d_j v_{0j} \quad \text{for } j=1, \cdots, m \tag{3-36}$$

Generally, we want to make S_x as large as possible. Since the set S_x becomes larger as the set S_z:

$$S_z = \{\underline{z} \mid \text{(3-35) and (3-36) are satisfied}\} \in R^n$$

is made larger, we can consider S_z instead of S_x. The set S_z is the inside of a hyperplane

$$\frac{\tilde{\pi}_1}{\tilde{\pi}_0} z_1 + \cdots + \frac{\tilde{\pi}_m}{\tilde{\pi}_0} z_m = 1 \qquad (3\text{-}37)$$

and the coordinate hyperplane $z_j = 0$ for $j=1, \cdots, m$. From the definition (3-31) and (3-36), $\tilde{\pi}_0/\tilde{\pi}_j = 1$ for some j and $\tilde{\pi}_0/\tilde{\pi}_j < 1$ for the other j's. This situation is illustrated in Fig. 3-1 for the case $m = 2$ and $\tilde{\pi}_1 < \tilde{\pi}_2$. Now, consider the normalized-vector $\underline{\pi}$;

$$\underline{\pi} = \frac{1}{\tilde{\pi}_1 + \cdots + \tilde{\pi}_m} \underline{\tilde{\pi}} \qquad (3\text{-}38)$$

which is orthogonal to the hyperplane (3-37) and terminates on the hyperplane

$$z_1 + \cdots + z_m = 1 \qquad (3\text{-}39)$$

Because of the property of S_z stated above and the relation of $\underline{\pi}$ and S_z we can expect that S_z becomes larger as the angle θ between $\underline{\pi}$ and the hyperplane (3-39) becomes nearer to the right angle (See Fig. 3-1). Since the terminal point of $\underline{\pi}$ is on the hyperplane (3-39), θ becomes nearer to the right angle if $||\underline{\pi}||$ becomes smaller. As a result we can expect that S_x becomes larger if $||\underline{\pi}||$ is made smaller. By the term " best estimate " of the stability region, we mean

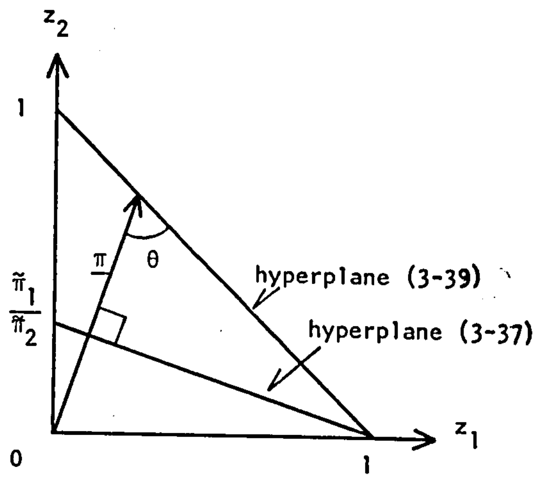

Fig. 3-1 Relation between $\underline{\pi}$ and S_z

the S_x which corresponds to the minimum value of $||\underline{\pi}||$

Now, let us consider how to find d_j's which make the value of $||\pi||$ smaller. The matrix $M^{1/2}DM^{1/2} - \tilde{\Gamma}^T K^{-1/2} DK^{-1/2}\tilde{\Gamma}$ can be expressed as

$$M^{1/2}DM^{1/2} - \tilde{\Gamma}^T K^{-1/2} DK^{-1/2}\tilde{\Gamma}$$

$$= \{\tilde{\pi}_1 + \cdots + \tilde{\pi}_m\} M^{1/2} V_0^{-1/2} (\Pi - \Theta^T \Pi \Theta) V_0^{-1/2} M^{1/2} \tag{3-40}$$

where $\Pi = \mathrm{diag}(\pi_j)$, $\Theta = V_0^{-1/2} K^{-1/2} \tilde{\Gamma} M^{-1/2} V_0^{1/2}$, $V_0 = \mathrm{diag}(v_{0j})$.

Since $\tilde{\pi}_1 + \cdots + \tilde{\pi}_m > 0$, the matrix of (3-40) is positive-semi-definite if and only if $\Pi - \Theta^T \Pi \Theta$ is positive-semi-definite. Therefore our problem turns out the problem of obtaining the vector $\underline{\pi}_0$ with the smallest norm $||\underline{\pi}_0||$ in the set

$$P = \{\underline{\pi} \mid \underline{\pi}_0 > 0 \ , \ \sum_{j=1}^{m} \pi_j = 1 \text{ and } \Pi - \Theta^T \Pi \Theta \text{ is positive-semi-definite for } \Pi = \mathrm{diag}(\pi_j)\} \tag{3-41}$$

If $I - \Theta^T\Theta$ is positive-semi-definite, the solution of the above minimization problems becomes staightforward, i.e. $\underline{\pi} = (1/m, \cdots, 1/m)^T$ is the vector with the smallest norm. If $I - \Theta^T\Theta$ is not positive-semi-definite, $||\underline{\pi}||$ takes the minimum value on the boundary of the set P (We can easily show that $||\underline{\pi}||$ cannot be the smallest on the open boundary $\pi_j = 0$ of P). In the latter case, we construct a subset Q of the set P and obtain the vector $\underline{\pi}$ with the smallest norm in Q. Such a vector can be determined by the method of Appendix IV of Demyanov & Malozemov (1974). The next theorem gives the set Q.

Theorem 3-4

Assume $I - \Theta$ is an M-matrix. Define the vectors $\underline{x}_j$ and $\underline{x}'_j$ (j=1, $\cdots$, m) by

$$\underline{x}_j = (I - \Theta)^{-1}\underline{e}_j \ , \quad \underline{x}'_j = (I - \Theta^T)^{-1}\underline{e}_j \tag{3-42}$$

where $e_j^{(j)} = 1$, $e_k^{(j)} = \varepsilon$, $j \neq k$; $j, k = 1, \cdots, m$ and $0 < \varepsilon < 1$.

Define the vectors $\underline{\pi}_{jk}$ (j, k = 1, $\cdots$, m) by

$$\underline{\pi}_{jk} = \tilde{\underline{\pi}}_{jk} \ / \ [\sum_{\ell=1}^{m} \tilde{\pi}_\ell^{(j,\,k)}] \tag{3-43}$$

$$\tilde{\pi}_\ell^{(j,\,k)} = x_\ell'^{(k)} / x_\ell^{(j)} \tag{3-44}$$

where $\tilde{\pi}_\ell^{(j,\,k)}$, $x_\ell'^{(k)}$ and $x_\ell^{(j)}$ are the ℓ-th components of the vectors $\tilde{\underline{\pi}}_{jk}$, $\underline{x}'_k$ and $\underline{x}_j$, respectively. Then, any member of the set Q defined by

$$Q = \{\underline{\pi} \mid \underline{\pi} = \sum_{j=1}^{m} \sum_{k=1}^{m} \delta_{jk}\underline{\pi}_{jk} \text{ where } \sum_{j=1}^{m} \sum_{k=1}^{m} \delta_{jk} = 1 \text{ and } \delta_{jk} \geqq 0 \} \tag{3-45}$$

is included in the set P. Namely, Q is the subset of P.

When $I - \Theta$ is irreducible, the above result is true for $\varepsilon = 0$.

Note that $I - \Theta$ is an M-matrix if and only if $K^{1/2}M^{1/2} - \tilde{\Gamma}$ is an M-matrix. (Lemma A-3).

The next lemma is a special case of Lemma A-4. It is stated here again with a brief proof because such is needed to prove Theorem 3-4.

Lemma 3-1

Since $I - \Theta$ is an M-matrix, there is a diagonal matrix $\Pi = \mathrm{diag}(\pi_j)$ with $\pi_j > 0$ such that $\Pi - \Theta^T\Pi\Theta$ is positive-definite.

[Proof] Since $I - \Theta$ is an M-matrix, there are vectors $\underline{x}$ and $\underline{x}'$ with positive constants such that the components of the vectors $\tilde{\underline{x}}'$ and $\tilde{\underline{x}}$;

$$\tilde{\underline{x}} = (I - \Theta)\underline{x}\ , \quad \tilde{\underline{x}}' = (I - \Theta)^T\underline{x} \tag{3-46}$$

are all positive. Determine $\underline{\pi}$ by

$$\pi_j = x_j'/x_j$$

where x_j and x_j' are the j-th components of $\underline{x}$ and $\underline{x}'$, respectively. Then,

$$\begin{aligned} &(\Pi - \Theta^T\Pi\Theta)\underline{x} \\ &= (I - \Theta)^T\Pi\underline{x} + \Theta^T\Pi(I - \Theta)\underline{x} \\ &= \tilde{\underline{x}}' + \Theta^T\Pi\tilde{\underline{x}} \end{aligned} \tag{3-47}$$

Evidently, the components of the vector on the righthand side are all positive. Therefore, $\Pi - \Theta^T\Pi\Theta$ is an M-matrix (Lemma A-1). Since $\Pi - \Theta^T\Pi\Theta$ is symmetric, it is positive-definite (Lemma A-1). [Q.E.D.]

[Proof of Theorem 3-4] All what we have to show is that $\Pi - \Theta^T\Pi\Theta$ becomes positive-semi-definite where $\Pi = \mathrm{diag}(\pi_j)$. When $I - \Theta$ is reducible, the components of $\underline{e}_j$ are all positive and, therefore, the components of $\underline{x}_j$ and $\underline{x}_j'$ are also all positive. So, $\underline{x}_j$ and $\underline{x}_j'$ satisfies the requirements on $\underline{x}$ and $\underline{x}'$ in the proof of Lemma 3-1. Therefore, $\Pi_{jk} = \mathrm{diag}(\pi_\ell^{(j,\ k)})$ $\ell=1, \cdots, m$ given by (3-43) make $\Pi - \Theta^T\Pi\Theta$ positive-definite for $\Pi = \Pi_{jk}$. Since $\Pi - \Theta^T\Pi\Theta$ is linear in Π, it remains positive-definite for Π being a linear combination of Π_{jk} with nonnegative coefficients; i.e. any member $\underline{\pi}$ of the set Q satisfies the requirement of Theorem 3-4.

When $I - \Theta$ is irreducible, we must step into the proof of Lemma 3-1. Since $I - \Theta$ is an irreducible M-matrix, the elements of $(I - \Theta)^{-1}$ are all positive (Lemma A-5). Therefore the components of $\underline{x}_j$ and $\underline{x}_j'$ are all positive. So, we can define $\Pi_{jk} = \mathrm{diag}(\pi_\ell^{(j,\, k)})$ by (3-44). Let $\Pi = \Pi_{jk}$ in the proof of Lemma 3-1. The righthand side of (3-47) becomes

$$\underline{e}_j + \Theta^T \Pi \underline{e}_j \quad \text{where } \Pi = \Pi_{jk} \text{ and } \varepsilon = 0.$$

The components of the above vector are non-negative. Therefore, the components of the vector

$$(\Pi - \Theta^T \Pi \Theta + \tau I)\underline{x}_j = \underline{e}_j + \Theta^T \Pi \underline{e}_j + \tau \underline{x}_j \quad \text{where } \Pi = \Pi_{jk}$$

are all positive for any $\tau > 0$. Therefore, the principal minors of $\Pi - \Theta^T \Pi \Theta$ are all non-negative. Since $\Pi - \Theta^T \Pi \Theta$ is symmetric, $\Pi - \Theta^T \Pi \Theta$ is positive-semi-definite for $\Pi = \Pi_{jk}$. The rest of the proof is the same with the reducible $I - \Theta$. [Q.E.D.]

In Example 3 of Chapter 7, we will compare the method given in this section with Weissenberger's method numerically.

Chapter 4. Stability Condition of Systems with Multiple Nonlinear Feedbacks

In this chapter, frequency-domain stability criteria of the multivariable nonlinear system of Fig. 1.1 are studied where the nonlinear feedback $\phi_j(\underline{y}, t)$ is assumed to satisfy $\zeta_j y_j^2 \leqq \phi_j(\underline{y}, t) y_j \leqq \eta_j y_j^2$.

Sec. 4.1. System Description and Well-Posedness

Let us consider the system described by

$$\dot{\underline{x}} = A\underline{x} + B\underline{e} \tag{4-1}$$

$$\underline{y} = C\underline{x} + D\underline{e} \tag{4-2}$$

$$\underline{e} = - \underline{\phi}(\underline{y}, t) + \underline{u} \tag{4-3}$$

where $\underline{x}$: n-vector, $\underline{y}$, $\underline{e}$, $\underline{u}$: m-vector, A : nxn constant matrix

B : nxm constant matrix, C : mxn constant matrix, D : mxm constant matrix

$\underline{\phi}(\underline{y}, t)$ is an m-vector whose elements are nonlinear time-varying functions:

$$\underline{\phi}(\underline{y}, t) = (\phi_1(\underline{y}, t), \cdots , \phi_m(\underline{y}, t))^T \tag{4-4}$$

$\phi_j(\underline{y}, t)$ satisfies

$$\zeta_j y_j^2 \leqq \phi_j(\underline{y}, t) y_j \leqq \eta_j y_j^2 \quad , \qquad \eta_j > 0 \tag{4-5}$$

where y_j is the j-th element of $\underline{y}$. In the following we refer to this system as the *system of type I* (simply as *System I*). The structure of System I is illustrated in Fig. 4.1 (, which is the same figure as Fig. 1.1) where the mxm transfer matrix $F(s) = (f_{jk}(s))$ is given by

$$F(s) = C(sI-A)^{-1}B + D \tag{4-6}$$

If we set $\underline{u}=\underline{0}$ in System I, the origin $\underline{x}=\underline{0}$ becomes an equilibrium. In the following, we are mainly concerned with Lyapunov stability of this equilibrium. Especially we investigate the next three cases.

Case 1 : $\zeta_1>0, \zeta_2>0, \cdots, \zeta_m>0$

Case 2 : $\zeta_1=0, \zeta_2=0, \cdots, \zeta_m=0$

Case 3 : $\zeta_1<0, \zeta_2<0, \cdots, \zeta_m<0$

Unfortunately, such cases as some ζ_j's are positive and the other ζ_j's are zero or negative cannot be treated in the present work. The conjugate transpose of a matrix Z is denoted by Z^*.

In Cases 1 and 3, we put

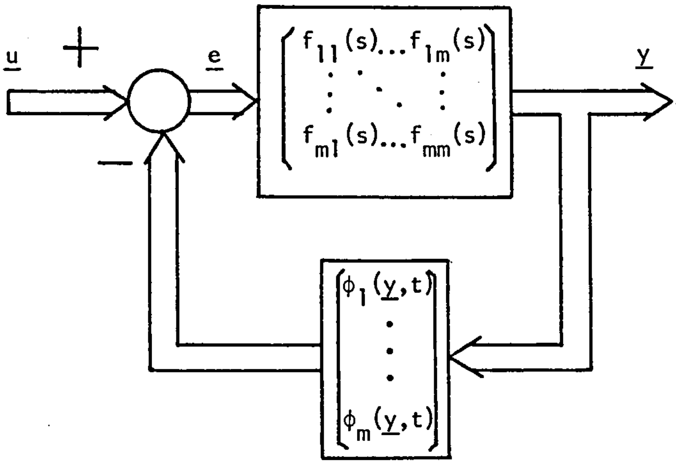

Fig. 4-1 Block diagram of System I

$$\zeta_j y_j^2 \leqq \phi_j(\underline{y}, t) y_j \leqq \eta_j y_j^2 \quad \text{for } j=1, \cdots, m$$

$$\kappa_j = \frac{1}{2}(\zeta_j^{-1} + \eta_j^{-1}), \quad r_j = \frac{1}{2}|\zeta_j^{-1} - \eta_j^{-1}| \tag{4-7}$$

The diagonal matrices with diagonal elements ζ_j, η_j, κ_j and r_j are denoted by ζ, η, K and R, respectively. We also take the existence of the matrices $[K + F(s)]^{-1}$ (in Case 1), $[\eta^{-1} + F(s)]^{-1}$ and $[\zeta^{-1} + F(s)]^{-1}$ (in Case 2 and Case 3) as granted. Actually, existence of such a matrix is a natural consequence of the facts that F(s) is the transfer matrix of the linear part (4-1) and that K, η, or ζ is nonsingular. To verify this we only need to multiply $Z^{-1} - Z^{-1}C(sI-A+BZ^{-1}C)^{-1}BZ^{-1}$ to $Z + C(sI-A)^{-1}B$ where Z is nonsingular.

Now, let us examine the well-posedness of System I. We assume that the nonlinear function ϕ_j satisfies not only (4-5) but also

$$|\phi_j(\underline{y}, t) - \phi_j(\underline{y}', t)| \leqq \beta_j(t)\ |y_j(t) - y'_j(t)| \tag{4-8}$$

where $\beta_j(t)$ is a piecewise continuous function and bounded in any finite interval. As System I has loops which consist of feed-through, by applying Theorem 2-3 we obtain the next sufficient condition of well-posedness.

Theorem 4-1 (Well-posedness condition of System I)

Let $\underline{\phi}$ satisfy (4-8). System I is well-posed in the sense of Definition 2-1, if one of the next two conditions is satisfied.

(i) $I-\tilde{B}(T)\tilde{D}$ is an M-matrix for all T where

$$(\tilde{D})_{jk} = |(D)_{jk}|, \quad \tilde{B}(T) = \mathrm{diag}(\beta_j^*(T)) \tag{4-9}$$

and

$$\beta_j^*(T) = \lim_{\Delta T \to 0} \sup_{t \in (T-\Delta T,\ T+\Delta T)} \beta_j(t) \qquad \text{for } j,\ k = 1, \cdots, m \tag{4-10}$$

(ii) $1-a \cdot b(T)$ is positive for all T where

$$a = \sqrt{\lambda_{max}(D^T D)}, \qquad b(T) = \max_{1 \leq j \leq m} \{\beta_j^*(T)\} \tag{4-11}$$

[Proof] (i) System I can be viewed as the concrete case of the CSF defined by (2-1) and (2-2) where $f_{jk}(s)$ and ϕ_j correspond to G_{jk} and H_{jj}, respectively ($H_{jk}=0$ for $j \neq k$). We consider the well-posedness in the L_2-sense, i.e. we set p=2. By applying Theorem 2-1 and Theorem 2-2, we obtain upper bounds of the uniform instantaneous gain of $f_{jk}(s)$ and ϕ_j as $|(D)_{jk}|$ and $\beta_j^*(T)$, respectively. So, the loop-gain-product matrix is $\tilde{B}(T)\tilde{D}$. By Theorem 2-3, System I is well-posed in the sense of Definition 2-1 if $I-\tilde{B}(T)\tilde{D}$ is an M-matrix for all T.

(ii) System I can also be viewed as a single-loop system which consists of two blocks G and H where G is the m-input m-output subsystem whose transfer matrix is F(s) and H is also the m-input m-output subsystem whose characteristics is given by $(H\underline{\sigma})(t) = (\phi_1(\underline{\sigma}, t), \cdots, \phi_m(\underline{\sigma}, t))^T$. Choose p=2. Then, by Theorem 2-1 the upper bound of the uniform instantaneous gain of G is a, and by Theorem 2-2 that of H is b(T). Since the loop-gain-product is $a \cdot b(T)$, System I is well-posed if $1-a \cdot b(T)$ is positive for all T. [Q.E.D.]

Sec. 4.2 Frequency Domain Conditions

Concerning the Lyapunov stability of System I, we have the next theorem.

Theorem 4-2 (Weighted multivariable circle criterion)

When $\underline{u}=\underline{0}$, the origin of System I is stable in the sense of Lyapunov if a diagonal matrix $W = \mathrm{diag}(w_j)$ with $w_j>0$ exists and the following condition is satisfied in each case.

(i) In Case 1, $[K + F(s)]^{-1}$ has no poles in the closed right-half plane Re s ≥ 0 and the matrix

$$Q(\omega) = [K+F(i\omega)]^* W^2 [K+F(i\omega)] - RW^2R \tag{4-12}$$

is positive-semi-definite for all real ω.

(ii) In Case 2, F(s) has no poles in the open right-half plane Re s >0 and the next three conditions are satisfied on the imaginary axis :

(a) The poles of $F(s)$ on the imaginary axis are simple.

(b) For each pole $i\omega_0$ on the imaginary axis, the residue of the matrix $\eta^{-1} + WF(s)W^{-1}$ defined by $\lim_{s \to i\omega_0} (s-i\omega_0)[\eta^{-1} + WF(s)W^{-1}]$ is Hermitian and positive-semi-definite.

(c) For real ω, the matrix

$$Q(\omega) = [\eta^{-1}+F(i\omega)]^* W^2 + W^2[\eta^{-1}+F(i\omega)] \tag{4-13}$$

is positive-semi-definite except at the poles in the imaginary axis.

(iii) In Case 3, $F(s)$ has no poles in the closed right-half plane Re $s \geqq 0$ and the matrix

$$Q(\omega) = RW^2R - [K+F(i\omega)]^* W^2[K+F(i\omega)] \tag{4-14}$$

is positive-semi-definite for real ω.

The condition that $[K + F(s)]^{-1}$ has no poles in Re $s \geqq 0$ (in Case 1) can be tested by various ways. For instance, when A, B and C of the linear system (4-1) are given, we can use the fact that the poles of $[K + F(s)]^{-1}$ are the eigenvalues of $A-BK^{-1}C$. When the transfer matrix $F(s)$ is given, we can use the Nyquist-type criteria included in Araki & Nwokah (1975). For this purpose, set f_j of the reference (Araki & Nwokah 1975) equal to $1/\kappa_j$. Then by the formulus (12) of Araki & Nwokah (1975), we can conclude that $[K + F(s)]^{-1}$ has no poles in Re $s \geqq 0$ if and only if the Nyquist diagram of $\det[K + F(s)]$ encircles the origin anti-clockwise as many times as the number of the poles of $F(s)$ in Re $s \geqq 0$ (Note that, when $F(s)$ has poles on the imaginary axis, the Nyquist contour must be intended into the left-half plane to avoid such poles (Araki & Nwokah 1975)). The test of positive-semi-definiteness of $Q(\omega)$ over $\omega=0 \sim \infty$ is not difficult with the aid of a computer if m is not very large. But it should be noted that arbitrary parameters w_j are included in the theorem and that this causes some inconvenience in applications. Actually, if we want to reach the sharpest result that can be derived from this theorem, we need to obtain a (practically testable) necessary and sufficient condition for the existence of W that satisfies the requirement of the theorem. But such a condition is not available at present (and seems not to appear in near future except for very special cases).

So, concerning determination of W, we must choose one of the following three alternative ways; to test the conditions (i), (ii), or (iii) for several a priori fixed W's, to use a sufficient (but not necessary) condition for the existence of W, or to facilitate a method to search a feasible W for a given system. If we choose the first or second way, we would be able to obtain a comparatively simple condition but have to be ready to miss better results. If we choose the third way, we would have to follow a little complicated procedure but be able to expect a sharper result.

Some of the previously reported criteria can be regarded as belonging to one of the three categories mentioned above though they were derived from seperate viewpoints. In reality, Rosenbrock's four criteria (1973) belong to the first category. The M-matrix condition of Araki (1976a) is a sufficient condition for the existence of W as shown in 4.3.3 and, so, belongs to the second.

We can state the condition of the theorem more compactly in terms of positive reality of matrices. In the next section such an alternative expression (i.e. Lemma 4-3) of the weighted multivariable circle criterion will be given. The reason why we choose the above as standard is that, from this form, the relation to the M-matrix condition can be easily seen and, from the relation, we are hinted about how to search a feasible W for a given system.

Theorem 4-3 (M-matrix condition)

There exists a constant diagonal matrix $W = \mathrm{diag}(w_j)$ with $w_j > 0$ that satisfies the requirement of Theorem 4-2 (and, hence, the origin of System I is stable in the sense of Lyapunov for the null input) if the following condition is satisfied in each case.

(i) In Case 1, $[K + F(s)]^{-1}$ has no poles in Re $s \geqq 0$ and the matrix $\Gamma - R$ is an M-matrix where $\Gamma = (\gamma_{jk})$ is given by

$$\begin{aligned} \gamma_{jj} &= \inf_{\omega} \ |\kappa_j + f_{jj}(i\omega)| \\ \gamma_{jk} &= -\sup_{\omega} \ |f_{jk}(i\omega)| \qquad (j \neq k) \end{aligned} \qquad (4\text{-}15)$$

(ii) In Case 2, F(s) has no poles in Re $s \geqq 0$ and the matrix Γ is an M-matrix where $\Gamma = (\gamma_{jk})$ is given by

$$\gamma_{jj} = \inf_{\omega} \ \mathrm{Re}[\eta_j^{-1} + f_{jj}(i\omega)] \qquad (4\text{-}16)$$

$$\gamma_{jk} = - \sup_{\omega} |f_{jk}(i\omega)| \qquad (j \neq k)$$

(iii) In Case 3, F(s) has no poles in Re s $\geqq 0$ and the matrix $R-\Gamma$ is an M-matrix where $\Gamma = (\gamma_{jk})$ is given by

$$\begin{aligned} \gamma_{jj} &= \sup_{\omega} |\kappa_j + f_{jj}(i\omega)| \\ \gamma_{jk} &= \sup_{\omega} |f_{jk}(i\omega)| \qquad (j \neq k) \end{aligned} \tag{4-17}$$

In the above theorem, the condition that $[K + F(s)]^{-1}$ has no poles in Re s $\geqq 0$ (in Case 1) can be assured by means of the Nyquist diagrams of $f_{jj}(s)$ on the main diagonal of F(s) as follows.

Theorem 4-4

In Case 1, the matrix $[K + F(s)]^{-1}$ has no poles in Re s $\geqq 0$ if the following three conditions are satisfied.

(a) F(s) has p_0 poles in Re s $\geqq 0$.

(b) The Nyquist diagram of $f_{jj}(s)$ encircles the point $(-\kappa_j, 0)$ p_j times anti-clockwise and $p_1 + \cdots + p_m = p_0$.

(c) The matrix Γ of (4-15) is an M-matrix.

The outline of the proof is as follows. If we set f_j of the reference (Rosenbrock 1972) equal to κ_j^{-1}, Theorem 2 of Rosenbrock (1972) is applicable to our problem. By the result of Araki & Nwokah (1975) we can show that, if the condition (c) holds true, the Nyquist diagram Γ_j of Theorem 2 of Rosenbrock (1972) encircles (-1,0) as many times as the Nyquist diagram of $f_{jj}(s)$ encircles $(-\kappa_j, 0)$. On account of this and the formula (12) of Rosenbrock (1972) we can conclude that $[K + F(s)]^{-1}$ has no poles in Re s $\geqq 0$.

In the above theorems note that Γ given by (4-15) becomes automatically an M-matrix if $\Gamma - R$ is so. Therefore, if we use Theorem 4-3 together with Theorem 4-4, the conditions to be tested in Case 1 are (a) and (b) of Theorem 4-4 and the one that $\Gamma - R$ is an M-matrix. If we take this fact into consideration we can immediately find that the stability condition given in Corollary 4-1 of Araki (1976a) is the same with that given in our Theorem 4-3 when F(s) is asymptotically stable. In Araki (1976a) L_2-stability is treated whereas Lyapunov stability is studied in our case. Since L_2-stability generally implies Lyapunov stability under certain controllability and observability assumption (J.C. Willems 1971), the above coincidence is not to be wondered. However, to have proved exact coincidence is significant.

Sec. 4.3 Proof of Theorems 4-2 and 4-3

4.3.1 Proof of Theorem 4-2

Six lemmas are presented before the proof of Theorem 4-2.

Lemma 4-1

When $\underline{u}=\underline{0}$ and $\zeta=0$, the origin of System I is stable in the sense of Lyapunov if a constant diagonal matrix $\hat{W} = \mathrm{diag}(\hat{w}_j)$ with $\hat{w}_j > 0$ exists and $\hat{D}(s)$ is positive-real where

$$\hat{D}(s) = \hat{W}[I + F(s)\eta]\hat{W}^{-1} \tag{4-18}$$

[Proof] As $\hat{D}(s) = I+\hat{W}D\hat{W}^{-1}\eta+\hat{W}C(sI-A)^{-1}B\hat{W}^{-1}\eta$ is assumed to be positive-real, by Anderson's lemma (1967) there exists a positive-definite matrix P and matrices L and H such that

$$PA + A^TP = -LL^T$$
$$PB\hat{W}^{-1}\eta - C^T\hat{W} = -LH$$
$$2I + \hat{W}D\hat{W}^{-1}\eta + \eta\hat{W}^{-1}D^T\hat{W} = H^TH$$

From the above equations we obtain the next inequality.

$$[\underline{x}^T,\ \underline{e}^T\hat{W}\eta^{-1}] \begin{bmatrix} PA+A^TP & PB\hat{W}^{-1}\eta-C^T\hat{W} \\ \eta\hat{W}^{-1}B^TP-\hat{W}C & -2I-\hat{W}D\hat{W}^{-1}\eta-\eta\hat{W}^{-1}D^T\hat{W} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \eta^{-1}\hat{W}\underline{e} \end{bmatrix}$$
$$= \underline{x}^T(PA + A^TP)\underline{x} + 2\underline{x}^TPB\underline{e} - 2\underline{x}^TC^T\hat{W}^2\eta^{-1}\underline{e} - 2\underline{e}^T\hat{W}^2\eta^{-2}\underline{e} - 2\underline{e}^T\hat{W}^2\eta^{-1}D\underline{e}$$
$$\leq 0 \tag{4-19}$$

We adopt a positive-definite quadratic form $v = \underline{x}^TP\underline{x}$ as the Lyapunov function candidate, then the time derivative along the solution of System I becomes

$$\dot{v}|_{\text{System I}} = \underline{x}^T(PA + A^TP)\underline{x} + 2\underline{x}^TPB\underline{e}$$

From (4-19)

$$\leq 2(\eta\underline{y} - \underline{\phi}(\underline{y}))^T\eta^{-2}\hat{W}^2\underline{\phi}(\underline{y})$$
$$= 2\sum_{j=1}^{m} (\eta_jy_j - \underline{\phi}_j(\underline{y}))\underline{\phi}_j(\underline{y})\hat{w}_j^2\eta_j^{-2}$$

From (4-5)

$$\dot{v}|_{\text{System I}} \leqq 0$$

As the time derivative along the solution of System I is shown to be negative-semi-definite, v is a valid Lyapunov function. Therefore the origin of System I with $\zeta=0$ and $\underline{u}=\underline{0}$ is stable in the sense of Lyapunov. [Q.E.D.]

Lemma 4-2 (Alternative form of the weighted multivariable circle criterion)

When $\underline{u}=\underline{0}$, the origin of System I is stable in the sense of Lyapunov if a constant diagonal matrix $\hat{W} = \mathrm{diag}(\hat{w}_j)$ with $\hat{w}_j>0$ exists and $\hat{D}(s)$ is positive-real where

$$\hat{D}(s) = \begin{cases} \hat{W}[\zeta^{-1} + F(s)]^{-1}[\eta^{-1} + F(s)]\hat{W}^{-1} & \text{Case 1} \quad (4\text{-}20) \\ \hat{W}[\eta^{-1} + F(s)]\hat{W}^{-1} & \text{Case 2} \quad (4\text{-}21) \\ -\hat{W}[\zeta^{-1} + F(s)]^{-1}[\eta^{-1} + F(s)]\hat{W}^{-1} & \text{Case 3} \quad (4\text{-}22) \end{cases}$$

[Proof] When the lower bound ζ of the nonlinearity $\underline{\phi}$ is not zero, Lemma 4-1 cannot be applied directly. So, we will convert System I into the form of Fig. 4-2 where $\underline{\tilde{\phi}}(\underline{y}, t)$ is defined by

$$\underline{\tilde{\phi}}(\underline{y}, t) = \underline{\phi}(\underline{y}, t) - \zeta\underline{y} \qquad (4\text{-}23)$$

From (4-23), $\underline{\tilde{\phi}}(\ ,\ )$ satisfies

$$0 \leqq \tilde{\phi}_j(\underline{y}, t)y_j \leqq (\eta_j - \zeta_j)y_j^2$$

$$\tilde{\phi}_j(\underline{0}, t) = 0$$

and $\tilde{F}(s)$ is given by

$$\tilde{F}(s) = [I + F(s)\zeta]^{-1}F(s)$$

We obtain a state space representation of the converted system as

$$\begin{aligned} \underline{\dot{x}} &= \tilde{A}\underline{x} + B\underline{\tilde{e}} \\ \underline{y} &= \tilde{C}\underline{x} + D\underline{\tilde{e}} \qquad (4\text{-}24) \\ \underline{\tilde{e}} &= -\underline{\tilde{\phi}}(\underline{y}, t) \end{aligned}$$

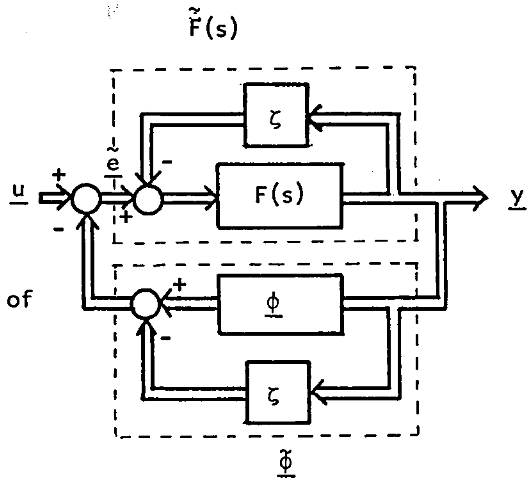

Fig. 4-2 Loop transformation of System I

where $\tilde{A}$=A-BζC and $\tilde{C}$=C-DζC. As the triplet [$\tilde{A}$, B, $\tilde{C}$] is completely controllable and observable, by applying Lemma 4-1 to the system given by (4-24) we obtain the stability condition; *When $\underline{u}=\underline{0}$, the origin of System I is stable in the sense of Lyapunov if a constant diagonal matrix $\hat{W} = \mathrm{diag}(\hat{w}_j)$ with $\hat{w}_j>0$ exixts and $\hat{D}(s)$ is* *positive-real where*

$$\hat{D}(s) = \hat{W}[I + \tilde{F}(s)(\eta - \zeta)]\hat{W}^{-1}$$
$$= \hat{W}[I + F(s)\zeta]^{-1}[I + F(s)\eta]\hat{W}^{-1} \tag{4-25}$$

We will clarify the relation between $\hat{D}(s)$ and $\hat{\hat{D}}(s)$ in each case.

In Case 1, as ζ, η and $\hat{W}$ are all positive-definite diagonal matrices, $\hat{D}(s)$ given by (4-25) can be represented as

$$\hat{D}(s) = \zeta^{-1}\hat{W}[\zeta^{-1}+F(s)]^{-1}[\eta^{-1}+F(s)]\hat{W}^{-1}\eta$$
$$= \zeta^{-1/2}\eta^{1/2}(\zeta^{-1/2}\eta^{-1/2})\hat{W}[\zeta^{-1}+F(s)]^{-1}[\eta^{-1}+F(s)]\hat{W}^{-1}(\zeta^{-1/2}\eta^{-1/2})\zeta^{-1/2}\eta^{1/2}$$

From this equality and Lemma 4-4, $\hat{D}(s)$ is positive-real if and only if

$$\zeta^{1/2}\eta^{-1/2}\hat{D}(s)\zeta^{1/2}\eta^{-1/2} = (\zeta^{-1/2}\eta^{-1/2})\hat{W}[\zeta^{-1}+F(s)]^{-1}[\eta^{-1}+F(s)]\hat{W}^{-1}(\zeta^{-1/2}\eta^{-1/2})$$

is positive-real. In this equality, if we put

$$\hat{\hat{W}} = \zeta^{-1/2}\eta^{-1/2}\hat{W}\ , \tag{4-26}$$

the righthand side of the above equality agrees with (4-20). Therefore if $\hat{\hat{D}}(s)$ given by (4-20) is positive-real, a diagonal matrix $\hat{W}$ given by (4-26) makes $\hat{D}(s)$ positive-real and, hence, System I is assured to be stable.

In Case 2, as ζ=0, we can immediately obtain (4-21) by putting

$$\hat{\hat{W}} = \hat{W} \tag{4-27}$$

in (4-25), Therefore, System I is stable, if $\hat{\hat{D}}(s)$ of (4-21) is positive-real.

In Case 3, as $-\zeta$, η and $\hat{W}$ are all positive-definite diagonal matrices, $\hat{D}(s)$ can be represented as

$$\hat{D}(s) = -(-\zeta)^{-1/2}\eta^{1/2}(-\zeta)^{-1/2}\eta^{-1/2}\hat{W}[\zeta^{-1}+F(s)]^{-1}[\eta^{-1}+F(s)]\hat{W}^{-1}(-\zeta)^{-1/2}\eta^{-1/2}$$
$$\cdot(-\zeta)^{-1/2}\eta^{1/2}\ ,$$

and from Lemma 4-4 $\hat{D}(s)$ is positive-real if and only if

$$(-\zeta)^{1/2}\eta^{-1/2}\hat{D}(s)(-\zeta)^{1/2}\eta^{-1/2} = -(-\zeta)^{-1/2}\eta^{-1/2}\hat{W}[\zeta^{-1}+F(s)]^{-1}[\eta^{-1}+F(s)]\hat{W}^{-1}$$
$$\cdot(-\zeta)^{-1/2}\eta^{-1/2}$$

is positive-real. In the above equation if we put

$$\hat{\hat{W}} = (-\zeta)^{-1/2}\eta^{-1/2}\hat{W}\ , \tag{4-28}$$

the righthand side of the equation agrees with (4-22). Therefore if $\hat{\hat{D}}(s)$ of (4-22) is positive-real, a diagonal matrix $\hat{W}$ given by (4-28) makes $\hat{D}(s)$ of (4-25) positive-real and, hence, System I is assured to be stable. This completes the proof of Lemma 4-2. [Q.E.D.]

To prove Theorem 4-2, we need the next four more lemmas about positive-real matrices.

Lemma 4-3

A real rational matrix $\hat{D}(s)$ is positive-real if and only if $\hat{D}(s)$ has no poles in the open right-half plane Re $s > 0$ and the next three conditions are satisfied on the imaginary axis:

(a) The poles of $\hat{\hat{D}}(s)$ on the imaginary axis are simple.

(b) For a pole $i\omega_0$ on the imaginary axis, the residue of $\hat{\hat{D}}(s)$ defined by $\lim_{s\to i\omega_0} (s-i\omega_0)\hat{\hat{D}}(s)$ is Hermitian and positive-semi-definite.

(c) For real ω, the matrix

$$\hat{\hat{D}}^*(i\omega) + \hat{\hat{D}}(i\omega) \tag{4-29}$$

is positive-semi-definite except at the poles on the imaginary axis.

Lemma 4-4

If Q is a real constant matrix and $\hat{D}(s)$ is positive-real, then $Q^T\hat{\hat{D}}(s)Q$ is also positive-real.

Lemma 4-5

$\hat{D}(s)$ is positive-real if and only if

$$E(s) = [\hat{D}(s) + I]^{-1}[\hat{D}(s) - I] \tag{4-30}$$

exists and is bounded-real.

Lemma 4-6

A real rational matrix E(s) is bounded-real if and only if E(s) has no poles in the closed right half plane and the matrix

$$I - E^*(i\omega)E(i\omega) \tag{4-31}$$

is positive-semi-definite for all real ω.

The proofs of the above four lemmas and the definition of bounded-reality are found in Newcomb (1966). Now, let us prove Theorem 4-2 using these lemmas.

[Proof of Theorem 4-2] First consider Case 1. By Lemma 4-5, $\hat{\hat{D}}(s)$ of (4-20) is positive-real if and only if

$$E(s) = \hat{W}[K + F(s)]^{-1}R\hat{W}^{-1}$$

is bounded-real. By Lemma 4-6, the above E(s) is bounded-real if and only if $[K + F(s)]^{-1}$ has no poles in Re $s \geq 0$ and the matrix

$$I-E^{*}(s)E(s) = I - \hat{W}^{-1}R[K+F^{*}(s)]^{-1}\hat{W}^{2}[K+F(s)]^{-1}\hat{W}^{-1}$$
$$= \hat{W}^{-1}R[K+F^{*}(s)]^{-1}\{[K+F^{*}(s)]R^{-1}\hat{W}^{2}R^{-1}[K+F(s)] - \hat{W}^{2}\}[K+F(s)]^{-1}R\hat{W}^{-1}$$

is positive-semi-definite for $s=i\omega$ where ω is real. If we note

$$\hat{W}^{-1}R[K+F^{*}(s)]^{-1} = \{[K+F(s)]^{-1}R\hat{W}^{-1}\}^{*}$$

we can conclude that the above condition agrees with the condition (i) of Theorem 4-2 for

$$W^{2} = R^{-1}\hat{W}^{2}R^{-1} \tag{4-32}$$

Next, consider Case 2. It is obvious from Lemma 4-3 that the condition (ii) of Theorem 4-2 agrees with the condition of Lemma 4-2 for

$$W = \hat{W} \tag{4-33}$$

Lastly, consider Case 3. The proof is parallel to that of Case 1 except that, in this case, E(s) is given by

$$E(s) = \hat{W}R^{-1}[K + F(s)]\hat{W}^{-1}$$

We can conclude that the condition (iii) of Theorem 4-2 agrees with the condition of Lemma 4-2 for

$$W^{2} = R^{-1}\hat{W}^{2}R^{-1} \tag{4-34}$$

In each case, it was shown that the condition of Lemma 4-2 is satisfied if and only if that of Theorem 4-2 is satisfied. Therefore, System I is assured to be stable in the sense of Lyapunov if Theorem 4-2 is satisfied. This completes the proof of Theorem 4-2. [Q.E.D.]

4.3.2 Remarks on Rosenbrock's Four Criteria

Concerning Lyapunov stability of System I in Case 1, Rosenbrock gave the next Theorem.

Theorem 4-5 (Rosenbrock 1973)

Assume $\zeta_1>0, \cdots, \zeta_m>0$. Then the origin of System I (The feed-through D is assumed to be zero) is stable in the sense of Lyapunov for the null input if any one of the next four matrices is positive-real.

$$D_1(s) = [\eta^{-1} + F(s)]^{-1}[\zeta^{-1} + F(s)]$$
$$D_2(s) = [\zeta^{-1} + F(s)][\eta^{-1} + F(s)]^{-1}$$

$$D_3(s) = [I + \eta F(s)][I + \zeta F(s)]^{-1}$$
$$D_4(s) = [I + F(s)\zeta]^{-1}[I + F(s)\eta]$$

If we study some examples we can immediately see that the four conditions in the theorem (i.e. positive-reality of the four theorems) are independent. Now, let us show that they all can be derived from the weighted multivariable circle criterion (Theorem 4-2) by setting W equal to certain values.

First, $D_4(s)$ is obtained from (4-25) by choosing $\hat{W} = I$. Therefore, from (4-26) and (4-32), positive-reality of $D_4(s)$ is equivalent that the condition (i) of Theorem 4-2 is satisfied for $W = R^{-1}\zeta^{-1/2}\eta^{-1/2}$.

Secondly, consider $D_1(s)$. By Lemma 4-5, $D_1(s)$ is positive-real if and only if

$$E_1(s) = [D_1(s) + I]^{-1}[D_1(s) - I]$$
$$= [K + F(s)]^{-1}R$$

is bounded-real. By Lemma 4-6, $E_1(s)$ is bounded-real if and only if $[K + F(s)]^{-1}$ has no poles in Re s >0 and the matrix

$$I - E_1^*(s)E_1(s) = R[K+F^*(s)]^{-1}\{[K+F^*(s)]R^{-1}R^{-1}[K+F(s)] - I\}[K+F(s)]^{-1}R \tag{4-35}$$

is positive-semi-definite for $s=i\omega$ where ω is real. It is easy to see that the matrix of (4-35) is positive-semi-definite for $s=i\omega$ if and only if $Q(\omega)$ of (4-12) is so for $W = R^{-1}$.

Thirdly, consider $D_2(s)$. We can show in the same way as the proof of $D_1(s)$. We can obtain that positive-reality of $D_2(s)$ is equivalent to the condition (i) of Theorem 4-2 with $W = I$.

Lastly, consider $D_3(s)$. Recall the fact that an invertible matrix is positive-real if and only if its inverse is positive-real (Newcomb 1966). Since $D_3(s)$ can be expressed as

$$D_3(s) = \eta[\eta^{-1} + F(s)][\zeta^{-1} + F(s)]^{-1}\zeta^{-1}$$

$D_3(s)$ is invertible (as a function of s) as noted at the end of Sec. 4.1. Therefore $D_3(s)$ is positive-real if and only if

$$[D_3(s)]^{-1} = \zeta[\zeta^{-1}+F(s)][\eta^{-1}+F(s)]^{-1}\eta^{-1}$$
$$= \zeta^{1/2}\eta^{-1/2}(\zeta^{1/2}\eta^{1/2})[\zeta^{-1}+F(s)][\eta^{-1}+F(s)]^{-1}(\zeta^{1/2}\eta^{1/2})^{-1}\zeta^{1/2}\eta^{-1/2}$$

is positive-real. From Lemma 4-4 and the proof of positive reality of $D_2(s)$, it is easy to see that $[D_3(s)]^{-1}$ is positive-real if and only if the condition (i) of Theorem 4-2 is satisfied for $W = \zeta^{1/2}\eta^{1/2}$.

We have proved that the four criteria of Rosenbrock are obtained from the weighted multivariable circle criterion by setting W according to $W = R^{-1}$, $W = I$, $W = \zeta^{1/2}\eta^{1/2}$, and $W = R^{-1}\zeta^{-1/2}\eta^{-1/2}$.

4.3.3 Lemmas on M-Matrices and Proof of Theorem 4-3

The fundamental properties of M-matrices are given in Appendix. The next three lemmas relate M-matrices to our main result (Theorem 4-2). The first two are easy consequences of the Lyapunov type theorems on M-matrices (Araki 1974). So, only brief proofs will be given. The second is the generalization of Lemma A-4 (or Lemma 3-1). The third was already reported in Araki (1974) in a little restricted form.

Lemma 4-7

If $\Gamma = (\gamma_{jk})$ is an M-matrix, there is a diagonal matrix $\tilde{W} = \mathrm{diag}(\tilde{w}_j)$ with $\tilde{w}_j > 0$ such that the Hermitian matrix

$$Q(H) = H^*\tilde{W} + \tilde{W}H \tag{4-36}$$

is positive-definite for any $H = (h_{jk})$ satisfying

$$\mathrm{Re}\, h_{jj} \geq \gamma_{jj}\ ,\quad |h_{jk}| \leq -\gamma_{jk}\quad (j \neq k) \tag{4-37}$$

[Proof] By Lemma A-1 there are vectors $\underline{x}$ and $\underline{x}'$ such that the elements of $\Gamma\underline{x}$ and $\Gamma^T\underline{x}'$ are all positive. Define a diagonal matrix $\tilde{W}$ by

$$\tilde{w}_j = x_j' \,/\, x_j \tag{4-38}$$

Then, the elements of $Q(\Gamma)\underline{x}$ become all positive. Since the off-diagonal elements of $Q(\Gamma)$ are non-positive it is an M-matrix by (ii) of Lemma A-1. Since (4-36) is positive-definite, the elements of $Q(\Gamma) = (\tilde{q}_{jk})$ and $Q(H) = (q_{jk})$ satisfy

$$q_{jj} \geq \tilde{q}_{jj}\ ,\quad |q_{jk}| \leq -\tilde{q}_{jk}\quad (j \neq k) \tag{4-39}$$

Therefore, Q(H) is also positive-definite.(Lemma A-2 is used in the last part. Note that this lemma holds true also when $\tilde{\Gamma}$ is an Hermitian complex matrix).

[Q.E.D.]

Lemma 4-8

Let $R = \text{diag}(r_j)$, $r_j > 0$, $\Gamma = (\gamma_{jk})$ and $\gamma_{jk} \geqq 0$. If $R - \Gamma$ is an M-matrix, there is a diagonal matrix $\tilde{W} = \text{diag}(\tilde{w}_j)$ with $\tilde{w}_j > 0$ such that the Hermitian matrix

$$Q(H) = R\tilde{W}R - H^*\tilde{W}H \tag{4-40}$$

is positive-definite for any $H = (h_{jk})$ satisfying

$$|h_{jk}| \leqq \gamma_{jk} \tag{4-41}$$

[Proof] By Lemma A-1 there are vectors $\underline{x}$ and $\underline{x}'$ such that the elements of $(I - \Gamma R^{-1})\underline{x}$ and $(I - \Gamma R^{-1})^T\underline{x}'$ are all positive. Determine $\tilde{W}$ by (4-38). Then we can conclude the positive-definiteness of Q(H) just in parallel to the proof of Lemma 4-7. [Q.E.D.]

Lemma 4-9

Let $\Gamma = (\gamma_{jk})$, $\gamma_{jk} > 0$ for $j \neq k$, $R = \text{diag}(r_j)$ and $r_j > 0$. If $\Gamma - R$ is an M-matrix, there is a diagonal matrix $\tilde{W} = \text{diag}(\tilde{w}_j)$ with $\tilde{w}_j > 0$ such that the Hermitian matrix

$$Q(H) = H^*\tilde{W}H - R\tilde{W}R \tag{4-42}$$

is positive-definite for any $H = (h_{jk})$ satisfying

$$|h_{jj}| \geqq \gamma_{jj}, \quad |h_{jk}| \leqq -\gamma_{jk} \quad (j \neq k) \tag{4-43}$$

[Proof] By Lemma A-1 there are vectors $\underline{x}$ and $\underline{x}'$ such that the elements of $(I - R\Gamma^{-1})\underline{x}$ and $(I - R\Gamma^{-1})^T\underline{x}'$ are all positive. Determine $\tilde{W}$ by (4-38). Then we can conclude the positive-definiteness of Q(H) just in parallel to the proof of Lemma 4-7. [Q.E.D.]

[Proof of Theorem 4-3] In Case 1, we obtain

$$|\kappa_j + f_{jj}(i\omega)| \geqq \gamma_{jj} \quad \text{and} \quad |f_{jk}(i\omega)| \leqq -\gamma_{jk} \quad (j \neq k)$$

from the definition of Γ. Therefore by Lemma 4-7, there is $\tilde{W} = \text{diag}(\tilde{w}_j)$ with $\tilde{w}_j > 0$ such that

$$Q(\omega) = [K+F(i\omega)]^*\tilde{W}[K+F(i\omega)] - R\tilde{W}R \tag{4-44}$$

is positive-definite for all ω. Determine W of Theorem 4-2 by

$$w_j = \tilde{w}_j^{1/2} \tag{4-45}$$

Then the matrix $Q(\omega)$ of (4-44) becomes equal to that of (4-12) and, hence, is positive-definite for all ω.

In Case 2, we obtain

$$\text{Re } [\eta_j^{-1} + f_{jj}(i\omega)] \geqq \gamma_{jj} \quad \text{and} \quad |f_{jk}(i\omega)| \leqq -\gamma_{jk} \quad (j \neq k)$$

from the definition of Γ. Therefore by Lemma 4-7, there is $\tilde{W} = \text{diag}(\tilde{w}_j)$ with $\tilde{w}_j > 0$ such that

$$Q(\omega) = [\eta^{-1} + F(i\omega)]^* \tilde{W} + \tilde{W}[\eta^{-1} + F(i\omega)] \qquad (4\text{-}46)$$

is positive-definite for all ω. Determine W of Theorem 4-2 by (4-45), then the matrix $Q(\omega)$ of (4-46) becomes equal to that of (4-13) and, hence, is positive-definite for all ω.

In Case 3, we obtain

$$\gamma_{jj} \geqq |\kappa_j + f_{jj}(i\omega)| \quad \text{and} \quad \gamma_{jk} \geqq |f_{jk}(i\omega)| \quad (j \neq k)$$

from the definition of Γ. Therefore, by Lemma 4-8, there is $\tilde{W} = \text{diag}(\tilde{w}_j)$ with $\tilde{w}_j > 0$ such that

$$Q(\omega) = R\tilde{W}R - [K+F(i\omega)]^* \tilde{W}[K+F(i\omega)] \qquad (4\text{-}47)$$

is positive-definite for all ω. Define W of Theorem 4-2 by (4-45), then the matrix $Q(\omega)$ of (4-46) becomes equal to that of (4-14) and, hence, is positive-definite for all ω. [Q.E.D.]

Sec. 4.4 Method to Search a Feasible Weight

As shown in the previous section, the weighted multivariable circle criterion is potential to give a sharper result than the Rosenbrock's criteria (Theorem 4-5) and the M-matrix condition (Theorem 4-3). However, to obtain such a good result, we must find an appropriate weight matrix W suited to a given system. In this section we will consider the problem of finding a "feasible" weight.

First, let us define a "feasible" weight. In Theorem 4-2, the weight $W^2 = \tilde{W}$ has influence only on the positive-semi-definiteness of $Q(\omega)$ except for the condition (ii-b) of Theorem 4-2. Since the norm of the weight $\tilde{W}$ does not affect the positive-semi-definiteness of $Q(\omega)$, without loss of generality we may consider only a normalized weight matrix $\tilde{W} = \text{diag}(\tilde{w}_j)$ which satisfies

$$\tilde{w}_1 + \cdots + \tilde{w}_m = 1 \qquad (4\text{-}48)$$

From these facts, it is natural to define a feasible weight as follows.

Definition 4-1

A positive-definite diagonal matrix $\tilde{W}$ is said to be *feasible* if $\tilde{W}$ satisfies

(4-48) and makes the matrix $Q(\omega)$ of Theorem 4-2 positive-semi-definite with $W^2 = \tilde{W}$ for all ω.

If one feasible weight exists for a given $Q(\omega)$, there are usually many other feasible weights, and they are the elements of a certain convex set. This is obvious from the fact that when $\tilde{W}_1$ and $\tilde{W}_2$ are feasible the weight given by $(\alpha\tilde{W}_1 + \beta\tilde{W}_2)/(\alpha + \beta)$ is also feasible for any positive numbers α and β. To represent the convex set explicitly, we define a set $V(\omega)$ as follows.

Definition 4-2

$V(\omega)$ is the set of positive-definite diagonal matrices $\tilde{W}$ which satisfy (4-48) and make $Q(\omega)$ of Theorem 4-2 positive-semi-definite with $W^2 = \tilde{W}$ for the value of ω, i.e.

$$V(\omega) = \{\tilde{W} = \mathrm{diag}(\tilde{w}_j) \mid \tilde{w}_j > 0,\ (4\text{-}48) \text{ is satisfied, and } Q(\omega) \geqq 0\}$$

where $Q(\omega)$ is given by

$$Q(\omega) = \begin{cases} [K+F(i\omega)]^*\tilde{W}[K+F(i\omega)] - R\tilde{W}R & \text{Case 1} \qquad (4\text{-}50) \\ [\eta^{-1}+F(i\omega)]^*\tilde{W} + \tilde{W}[\eta^{-1}+F(i\omega)] & \text{Case 2} \qquad (4\text{-}51) \\ R\tilde{W}R - [K+F(i\omega)]^*\tilde{W}[K+F(i\omega)] & \text{Case 3} \qquad (4\text{-}52) \end{cases}$$

Then, the convex set is given by the intersection of $V(\omega)$ for all ω, i.e. $\bigcap_\omega V(\omega)$. In the following, we consider the problem of obtaining a weight $\tilde{W}$ which belongs to the set $\bigcap_\omega V(\omega)$. It is not easy to determine $V(\omega)$ exactly, but we can obtain a subset of $V(\omega)$ by the next theorem.

Theorem 4-6

(i) In Case 1, assume that $\tilde{\Gamma}(\omega) - R$ is an M-matrix where $\tilde{\Gamma} = (\tilde{\gamma}_{jk}(\omega))$ and

$$\tilde{\gamma}_{jj}(\omega) = |\kappa_j + f_{jj}(i\omega)|, \quad \tilde{\gamma}_{jk}(\omega) = -|f_{jk}(i\omega)| \ (j\neq k) \qquad (4\text{-}53)$$
$$\text{for } j, k = 1, \cdots, m$$

Define the vectors $\underline{x}_j$ and $\underline{x}_j'$ $(j=1, \cdots, m)$ by

$$\underline{x}_j = \Delta^{-1}\underline{e}_j, \quad \underline{x}_j' = (\Delta^T)^{-1}\underline{e}_j \qquad (4\text{-}54)$$

where $\Delta = I - R\tilde{\Gamma}^{-1}(\omega)$ and $e_j^{(j)} = 1$, $e_j^{(k)} = \varepsilon$ $(j\neq k)$; $j, k = 1, \cdots, m$ and $0 < \varepsilon \leqq 1$. Define a diagonal matrix $\tilde{\tilde{W}}_{jk}$ $(j, k = 1, \cdots, m)$ by

$$\tilde{\tilde{W}}_{jk} = \overline{W}_{jk}/[\sum_{\ell=1}^{m} \overline{w}_\ell^{(j,k)}] \qquad (4\text{-}55)$$

$$\overline{W}_{jk} = \mathrm{diag}(\overline{W}_{\ell}^{(j,\ k)}) \text{ and } \overline{W}_{\ell}^{(j,\ k)} = x'^{(k)}_{\ell}/x^{(j)}_{\ell} \qquad (4\text{-}56)$$
$$\text{for } j,\ k,\ \ell = 1, \cdots, m$$

Then, the set

$$U(\omega) = \{\tilde{W} \mid \tilde{W} = \sum_{j,k=1}^{m} \alpha_{jk}\tilde{\tilde{W}}_{jk}(\omega) \text{ where } \alpha_{jk} \geqq 0 \text{ and } \sum_{j=1}^{m}\sum_{k=1}^{m} \alpha_{jk} = 1\} \qquad (4\text{-}57)$$

is a subset of $V(\omega)$.

When $\tilde{\Gamma}(\omega) - R$ is irreducible, the above result is true for $\varepsilon = 0$.

(ii) In Case 2, assume that $\tilde{\Gamma}(\omega)$ is an M-matrix where $\tilde{\Gamma}(\omega) = (\tilde{\gamma}_{jk}(\omega))$ and

$$\tilde{\gamma}_{jj} = 1/\eta_j + \mathrm{Re}\,\{f_{jj}(i\omega)\}, \quad \tilde{\gamma}_{jk} = -|f_{jk}(i\omega)| \ (j \neq k) \qquad (4\text{-}58)$$
$$\text{for } j,\ k = 1, \cdots, m$$

Define the matrix Δ in (4-54) by $\Delta = \tilde{\Gamma}(\omega)$ and a diagonal matrix $\tilde{\tilde{W}}_{jk}$ by (4-54), (4-55) and (4-56). Then, the set defined by (4-57) is a subset of $V(\omega)$.

When $\tilde{\Gamma}(\omega)$ is irreducible, the above result is true for $\varepsilon = 0$.

(iii) In Case 3, assume that $R - \tilde{\Gamma}(\omega)$ is an M-matrix where $\tilde{\Gamma}(\omega) = (\tilde{\gamma}_{jk}(\omega))$ and

$$\tilde{\gamma}_{jj}(\omega) = |\kappa_j + f_{jj}(i\omega)|\ ,\ \tilde{\gamma}_{jk}(\omega) = |f_{jk}(i\omega)| \quad (j \neq k) \qquad (4\text{-}59)$$
$$\text{for } j,\ k = 1, \cdots, m$$

Define the matrix Δ in (4-54) by $\Delta = I - \tilde{\Gamma}(\omega)R^{-1}$ and a diagonal matrix $\tilde{\tilde{W}}_{jk}$ by (4-54),(4-55) and (4-56). Then, the set $U(\omega)$ defined by (4-57) is a subset of $V(\omega)$.

When $R - \tilde{\Gamma}(\omega)$ is reducible, the above result is true for $\varepsilon = 0$.

We can prove this theorem in the same way as the proof of Theorem 3-4 by using Lemma 4-9, Lemma 4-7 and Lemma 4-8 in Case 1, Case 2, and Case 3, respectively, instead of Lemma 3-1. So, the proof of Theorem 4-6 is omitted.

Let us consider what value of ε of Theorem 4-6 should be taken. As we can expect that the subset $U(\omega)$ becomes larger as the value of ε becomes smaller, it is desireable to take the value of ε as small as possible. If the M-matrix is irreducible, we can choose $\varepsilon = 0$. If it is not so, we cannot choose so small positive number ε for the next reason. Consider the critical case that we choose $\varepsilon = 0$ in Case 1. When $\tilde{\Gamma}(\omega) - R$ is reducible, some of the elements of Δ^{-1} are zero and so some of the components of $\underline{x}_j$ and $\underline{x}_j'$ become zero. As a result, division by zero occurs for some triplet (j, k, ℓ) in (4-56). Therefore, we cannot choose ε so small that the round off error cannot be neglected.

It should be noted that the M-matrix requirements of Theorem 4-5 is much weaker than those of Theorem 4-3.

Now, let us consider how to use $U(\omega)$ to obtain a feasible weight. We think of the next two methods; i.e. to calculate the intersection of $U(\omega)$ for all ω strictly, or to search a weight which can be expected to belong to the set $\bigcap_{\omega} V(\omega)$ by using $U(\omega)$ only as an index of $V(\omega)$. If we apply the former method, we can surely obtain a feasible weight when $\bigcap_{\omega} U(\omega)$ is not null. It seems to be difficult and needs much computation to obtain the intersection of $U(\omega)$ for all ω. When $\bigcap_{\omega} U(\omega)$ is null, we cannot obtain a feasible weight by the former method, and we have to use the latter one. As the weight determined by this method is not assured to belong to $\bigcap_{\omega} V(\omega)$, it is necessary to ensure that the weight is feasible. From the result of some numerical examples, we can point out that in such cases as both the M-matrix condition and the Rosenbrock's criteria cannot assure stability, in other words, the cases in which the weighted multivariable circle criterion must be used, the set $\bigcap_{\omega} U(\omega)$ is usually null. This is mainly because in such cases the M-matrix condition of Theorem 4-6 is unsatisfied for some ω. In this case the former method always fails to obtain a feasible weight. Considering the above facts, we will investigate only the latter method.

We propose a method of searching a feasible weight by using $U(\omega)$ as an index of $V(\omega)$.

Procedure of searching a feasible weight $\tilde{W}$

Step 1) Obtain the interval of ω for which M-matrix condition of Theorem 4-6 is satisfied.

Step 2) Choose several frequencies $\omega = \omega_i$ $(i=1, \cdots, q)$ from the interval obtained in Step 1.

Step 3) For those $\omega = \omega_i$ $(i=1, \cdots, q)$, compute positive-definite diagonal matrices $\tilde{\tilde{W}}_{jk}(\omega_i)$ $(j, k = 1, \cdots, m)$ by (4-55).

Step 4) Compute $\tilde{W} = \mathrm{diag}(\tilde{w}_\ell)$ by

$$\tilde{w}_\ell = \frac{1}{2}\{ \min_i \max_{j,k} [\tilde{\tilde{W}}_{jk}(\omega_i)]_{\ell\ell} + \max_i \min_{j,k} [\tilde{\tilde{W}}_{jk}(\omega_i)]_{\ell\ell} \} \quad (4\text{-}60)$$

We choose this $\tilde{W}$ as a candidate for the feasible weight.

Let us explain this procedure. From Step 1 to Step 3, we compute the subset $U(\omega)$ for each frequencyes $\omega = \omega_i$ $(i=1, \cdots, q)$. Considering the facts that $V(\omega)$ and $U(\omega)$ are convex sets and $U(\omega)$ is a subset of $V(\omega)$, we can expect that the " center " of $\bigcap_{\omega} U(\omega)$ would be feasible, and in Step 4 the " center " of $\bigcap_{i} U(\omega_i)$ is given. Let us explain this in more detail. As $U(\omega_i)$ has the form of (4-58), each element $W(\omega_i)$ of the set $U(\omega_i)$ satisfies

$$\min_{j,k}[\tilde{\tilde{W}}_{jk}(\omega_i)]_{\ell\ell} \leq [W(\omega_i)]_{\ell\ell} \leq \max_{j,k}[\tilde{\tilde{W}}_{jk}(\omega_i)]_{\ell\ell} \text{ for } \ell = 1, \cdots, m$$

If the intersection of $U(\omega_i)$ is not null, each element W of $\bigcap_i U(\omega_i)$ satisfies

$$\max_i \min_{j,k}[\tilde{\tilde{W}}_{jk}(\omega_i)]_{\ell\ell} \leq [W]_{\ell\ell} \leq \min_i \max_{j,k}[\tilde{\tilde{W}}_{jk}(\omega_i)]_{\ell\ell} \text{ for } \ell = 1, \cdots, m$$

In (4-60), the center of this interval is calculated. Unfortunately the obtained weight is not guaranteed to belong to $\bigcap_i U(\omega_i)$. If the intersection is null, it is difficult to explain what this equation (4-60) means, but the following examples would be helpful to understand this equality intuitively. To visualize Step 4, let us consider m-dimensional vector space where the diagonal matrix $\tilde{W} = \mathrm{diag}(\tilde{w}_j)$ is represented as an m-vector $\underline{\tilde{w}} = (\tilde{w}_j)$. In this space, the set $U(\omega)$ corresponds to a hyperpolyhedron which is on a hyperplane (4-48). Three examples are illustrated in Fig. 4-3 where q=3 and m=3. In this figure, the hyperpolyhedrons are projected onto the $\tilde{w}_1$-$\tilde{w}_2$ plane and the symbol x corresponds to $\tilde{W}$ given by (4-60). This method is not strict, but this method is practical in the point that the computation is simple and that in many cases this method gives a feasible weight whereas the strict method fails to do so.

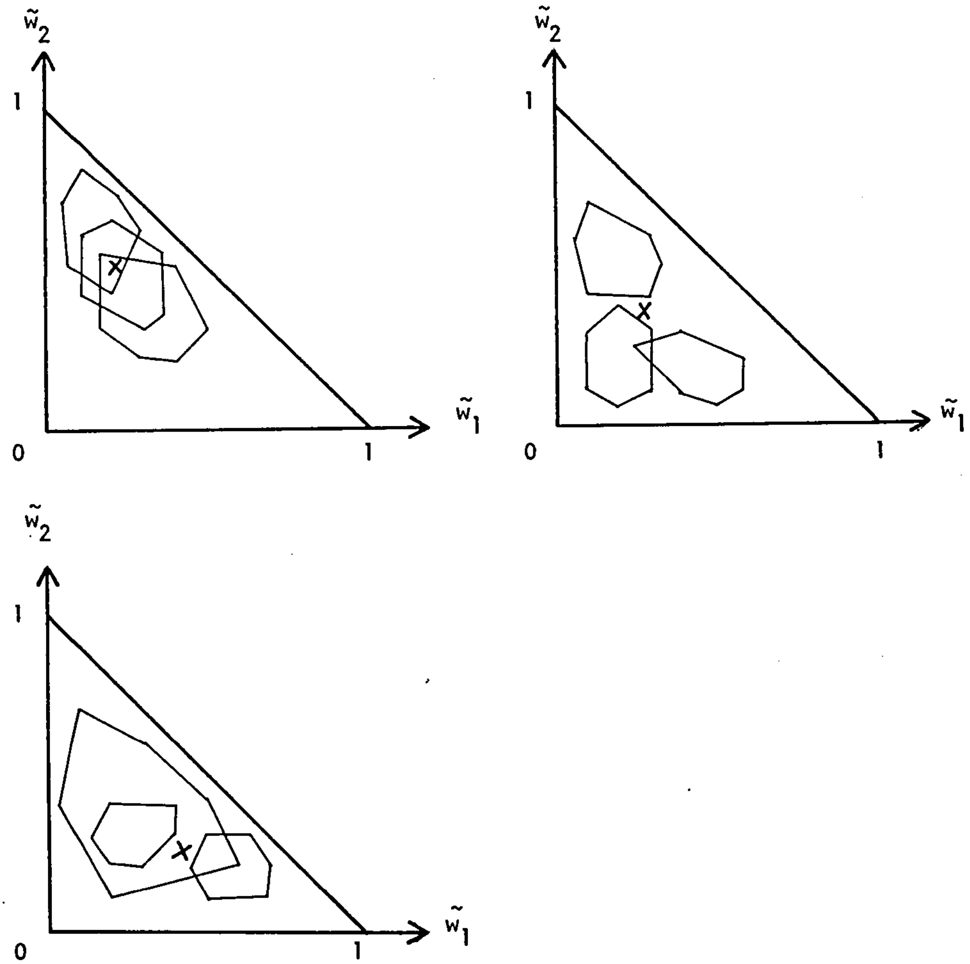

Fig. 4-3 Examples of searching a feasible $\tilde{W}$
x is determined by (4-61) and hexagons
denote the sets U_j (j=1,2,3)

Chapter 5 Stability Condition of Composite System Consisting of Subsystems with Nonlinear Feedbacks

In this chapter, the composite system given in Fig. 1.2 is classified into four cases according to the subsystem assumptions and in each case the Lyapunov stability condition is obtained. The relation between these conditions and the L_2-stability conditions are clarified.

Sec. 5.1. Classification of Composite Systems and Well-Posedness

We investigate a certain class of composite systems which have the following characteristics.

- Each subsystem is Lur'e type system which is familiar to control engineers through the absolute stability problem.
- These subsystems are interconnected from their outputs to their inputs as shown in Fig. 1.2.

We refer to this class of systems as the *system of type II* (simply as *System II*)

Let us classify System II into the next four cases according to the subsystem assumptions. In each case, we assume that the interconnection functions are linearly bounded and time-varying nonlinearities.

Case A: Subsystems are all single-input single-output with nonlinear time-varying feedbacks $\phi_j(y_j, t)$ satisfying $\zeta_j y_j^2 \leqq \phi_j y_j \leqq \eta_j y_j^2$.

Case B: Subsystems are all single-input single-output with nonlinear time-invariant feedbacks $\phi_j(y_j)$ satisfying $0 \leqq \phi_j y_j \leqq \eta_j y_j^2$.

Case C: Subsystems are all multi-input multi-output with nonlinear time-varying feedbacks $\underline{\phi}_j(\underline{y}_j, t)$ satisfying $\zeta_k^{(j)}(y_k^{(j)})^2 \leqq \phi_k^{(j)} y_k^{(j)} \leqq \eta_k^{(j)}(y_k^{(j)})^2$.

Case D: Subsystems are all multi-input multi-output with nonlinear time-invariant feedbacks $\underline{\phi}_j(\underline{y}_j)$ satisfying $0 \leqq \phi_k^{(j)} y_k^{(j)} \leqq \eta_k^{(j)}(y_k^{(j)})^2$.

Now, let us describe the above four cases precisely.

In Case A, we consider a composite system described by

$$\dot{\underline{x}}_j = A_j \underline{x}_j + \underline{b}_j\{-\phi_j(y_j, t) + z_j\} \tag{5-1}$$

$$y_j = \underline{c}_j^T \underline{x}_j \tag{5-2}$$

$$z_j = g_j(y_1, \cdots, y_m, t) + \tilde{z}_j \qquad j=1, \cdots, m \quad (5\text{-}3)$$

where y_j, z_j, $\tilde{z}_j$, ϕ_j and g_j are scalars and $\underline{x}_j$, $\underline{c}_j$, $\underline{b}_j$ are m_j-vectors. The scalar valued functions $\phi_j(y_j, t)$ and $g_j(y_1, \cdots, y_m, t)$ are assumed to satisfy

$$\zeta_j y_j^2 \leqq \phi_j(y_j, t) y_j \leqq \eta_j y_j^2 \qquad (5\text{-}4)$$

$$\phi_j(0, t) = 0 \qquad (5\text{-}5)$$

$$|g_j(y_1, \cdots, y_m, t)| \leqq \sum_{k=1}^{m} \beta_{jk} y_k \quad \text{for } j=1, \cdots, m \qquad (5\text{-}6)$$

where β_{jk} is a non-negative constant, η_j is a positive constant and ζ_j is a constant less than η_j.

We refer to this system (5-1)-(5-6) as *System IIA.*

In Case B, we consider a composite system described by

$$\dot{\underline{x}}_j = A_j \underline{x}_j + \underline{b}_j\{-\phi_j(y_j) + z_j\} \qquad (5\text{-}7)$$

$$y_j = \underline{c}_j^T \underline{x}_j \qquad (5\text{-}8)$$

$$z_j = g_j(y_1, \cdots, y_m, t) + \tilde{z}_j \quad j=1, \cdots, m \qquad (5\text{-}9)$$

where y_j, z_j, $\tilde{z}_j$, ϕ_j and g_j are scalars and $\underline{x}_j$, $\underline{c}_j$, $\underline{b}_j$ are m_j-vectors. The scalar-valued functions $\phi_j(y_j)$ and $g_j(y_1, \cdots, y_m, t)$ are assumed to satisfy

$$0 \leqq \phi_j(y_j) y_j \leqq \eta_j y_j^2 \qquad (5\text{-}10)$$

$$\phi_j(0) = 0 \qquad (5\text{-}11)$$

$$|g_j(y_1, \cdots, y_m, t)| \leqq \sum_{k=1}^{m} \beta_{jk} |y_k| \quad \text{for } j=1, \cdots, m \qquad (5\text{-}12)$$

We refer to this system (5-7)-(5-12) as *System IIB.*

In case C, we consider a composite system described by

$$\dot{\underline{x}}_j = A_j \underline{x}_j + B_j\{-\underline{\phi}_j(\underline{y}_j, t) + \underline{z}_j\} \qquad (5\text{-}13)$$

$$\underline{y}_j = C_j \underline{x}_j \qquad (5\text{-}14)$$

$$\underline{z}_j = \underline{g}_j(\underline{y}_1, \cdots, \underline{y}_m, t) + \tilde{\underline{z}}_j \qquad j=1, \cdots, m \quad (5\text{-}15)$$

where $\underline{y}_j$: s_j-vector, $\underline{z}_j$: s_j-vector, $\tilde{\underline{z}}_j$: s_j-vector, $\underline{x}_j$: ℓ_j-vector

A_j : $\ell_j \times \ell_j$ constant matrix, B_j : $\ell_j \times s_j$ constant matrix

C_j : $s_j \times \ell_j$ constant matrix

$\underline{g}_j$: s_j-vector , $\underline{\phi}_j$: s_j-vector

The vector valued function

$$\underline{\phi}_j(\underline{y}_j, t) = (\phi_1^{(j)}(\underline{y}_j, t), \cdots, \phi_{s_j}^{(j)}(\underline{y}_j, t))^T \tag{5-16}$$

is assumed to satisfy

$$\zeta_k^{(j)}(y_k^{(j)})^2 \leqq \phi_k^{(j)}(\underline{y}_j, t)y_k^{(j)} \leqq \eta_k^{(j)}(y_k^{(j)})^2 \tag{5-17}$$

$$\underline{\phi}_j(\underline{0}, t) = \underline{0} \tag{5-18}$$

for $k=1, \cdots, s_j$, $j=1, \cdots, m$

where $y_k^{(j)}$ denotes the k-th element of a vector $\underline{y}_j$, $\eta_k^{(j)}$ are positive constants and $\zeta_k^{(j)}$ is a constant less than $\eta_k^{(j)}$. The vector valued function $\underline{g}_j(\underline{y}_1, \cdots, \underline{y}_m, t)$ is assumed to satisfy

$$|\underline{g}_j(\underline{y}_1, \cdots, \underline{y}_m, t)| \leqq \sum_{k=1}^{m} \beta_{jk}|\underline{y}_k| \tag{5-19}$$

where β_{jk} is an $s_j \times s_k$ constant matrix whose elements are nonnegative, $|\underline{y}_k|$ is an s_k - vector defined by

$$|\underline{y}_k| = (|y_1^{(k)}|, \cdots, |y_{s_k}^{(k)}|)^T \tag{5-20}$$

and $|\underline{g}_j|$ is an s_j - vector defined by

$$|\underline{g}_j| = (|g_1^{(j)}|, \cdots, |g_{s_j}^{(j)}|)^T \tag{5-21}$$

The inequality (5-19) is satisfied elementwise.

We refer to this system (5-13)-(5-19) as *System IIC*.

In Case D, we consider a composite system described by

$$\dot{\underline{x}}_j = A_j\underline{x}_j + B_j\{-\underline{\phi}_j(\underline{y}_j) + \underline{z}_j\} \tag{5-22}$$

$$\underline{y}_j = C_j\underline{x}_j \tag{5-23}$$

$$\underline{z}_j = \underline{g}_j(\underline{y}_1, \cdots, \underline{y}_m, t) + \tilde{\underline{z}}_j \quad j=1, \cdots, m \tag{5-24}$$

where the orders of the vectors and matrices are the same as those of Case C.

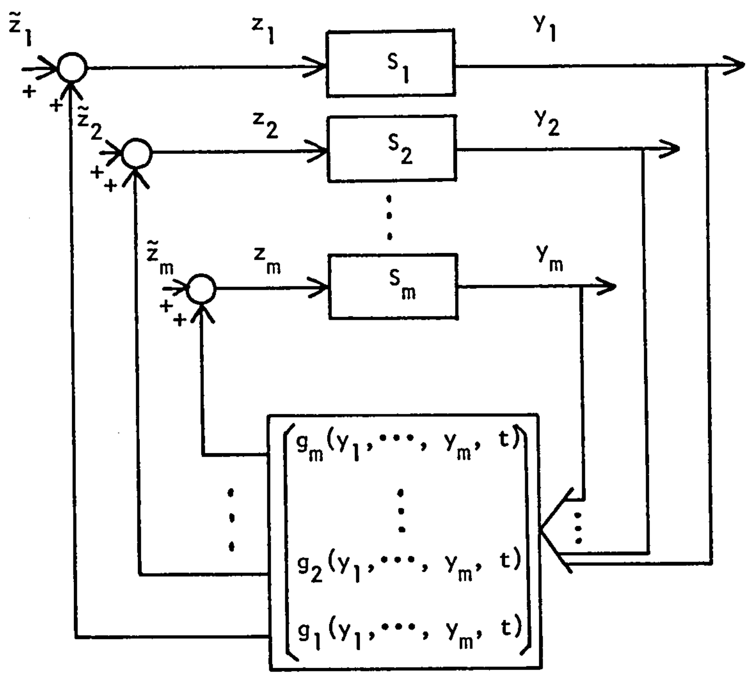

Fig. 5-1 (a) Block diagram of System IIA and System IIB

S_j is a single-input single-output subsystem.

$|g_j(y_1, \cdots, y_m, t)| \leqq \sum_{k=1}^{m} \beta_{jk}|y_k|$

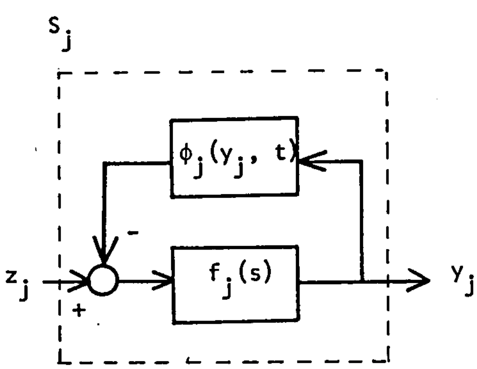

Fig. 5-1 (b)

Subsystem S_j of System IIA

$\zeta_j y_j^2 \leqq \phi_j(y_j, t)y_j \leqq \eta_j y_j^2$

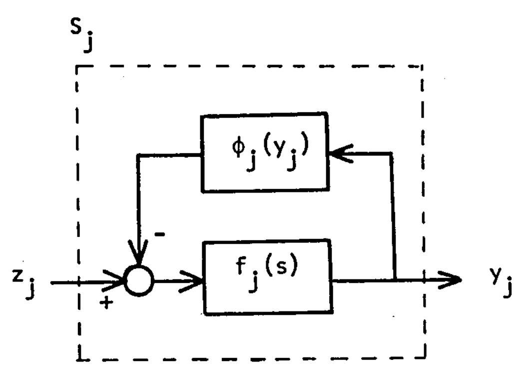

Fig. 5-1 (c)

Subsystem S_j of System IIB

$0 \leqq \phi_j(y_j)y_j \leqq \eta_j y_j^2$

The vector valued function $\underline{\phi}_j(\underline{y}_j)$ given by

$$\underline{\phi}_j(\underline{y}_j) = (\underline{\phi}_1^{(j)}(\underline{y}_j), \cdots, \underline{\phi}_{s_j}^{(j)}(\underline{y}_j))^T \tag{5-25}$$

is assumed to satisfy

$$0 \leqq \phi_k^{(j)}(\underline{y}_j)y_k^{(j)} \leqq \eta_k^{(j)}(y_k^{(j)})^2 \tag{5-26}$$

$$\underline{\phi}_j(\underline{0}) = \underline{0} \tag{5-27}$$

$$\text{for } j=1, \cdots, m, \ k=1, \cdots, s_j$$

where $y_k^{(j)}$ is the k-th component of $\underline{y}_j$ and $\eta_k^{(j)}$ is a positive constant. The vector valued function $\underline{g}_j(\underline{y}_1, \cdots, \underline{y}_m, t)$ satisfies

$$|\underline{g}_j(\underline{y}_1, \cdots, \underline{y}_m, t)| \leqq \sum_{k=1}^{m} \beta_{jk}|\underline{y}_k| \tag{5-28}$$

This inequality is the same as (5-19).

We refer to this system (5-22)-(5-28) as *System IID.*

Remark 5-1

The block diagram of System IIA is illustrated in Fig. 5-1 (a) and each subsystem S_j is illustrated in Fig. 5-1 (b). The transfer function $f_j(s)$ is given by

$$f_j(s) = \underline{c}_j^T(sI - A_j)^{-1}\underline{b}_j \tag{5-29}$$

$\underline{z}_j$ is the subsystem input, $\tilde{\underline{z}}_j$ is the external input, and $\underline{y}_j$ is the output. $\underline{g}_j$ interconnects subsystems. Let $\underline{x}$ be the state vector of this system given by

$$\underline{x} = (\underline{x}_1^T, \cdots, \underline{x}_m^T)^T \tag{5-30}$$

Then, from (5-5) and (5-12), the origin $\underline{x} = \underline{0}$ is an equilibrium point for the null input $\tilde{\underline{z}} = (\tilde{\underline{z}}_1, \cdots, \tilde{\underline{z}}_m)^T = \underline{0}$.

Remark 5-2

System IIB has the same structure as System IIA (Fig. 5-1 (a)) and each subsystem is illustrated in Fig. 5-1 (c). System IIB is a special case of System IIA and Remark 5-1 holds in this case, too. The condition (5-10) that the lower bound of $\phi_j y_j$ is zero is not restrictive. Because if the lower bound of $\phi_j y_j$ is ζ_j ($\neq 0$), we can convert the system into the form (5-7)-(5-12), in

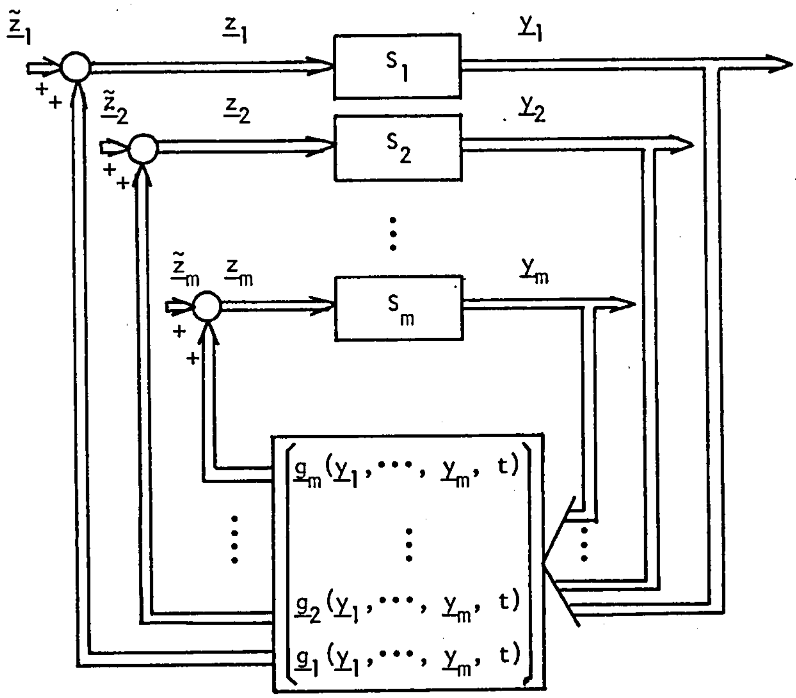

Fig. 5-2 (a) Block diagram of System IIC and System IID

S_j is a s_j-input s_j-output subsystem

$|g_j(y_1, \cdots, y_m, t)| \leqq \sum_{k=1}^{m} \beta_{jk} |y_k|$

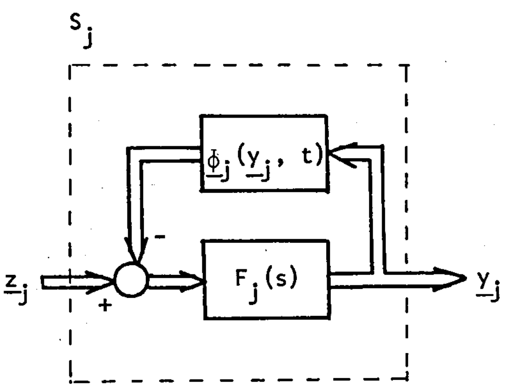

Fig. 5-2 (b)

Subsystem S_j of System IIC

$\zeta_k^{(j)}[y_k^{(j)}]^2 \leqq \phi_k^{(j)}(y_j)y_k^{(j)} \leqq \phi_k^{(j)}[y_k^{(j)}]^2$

for $k=1, \cdots, s_j$

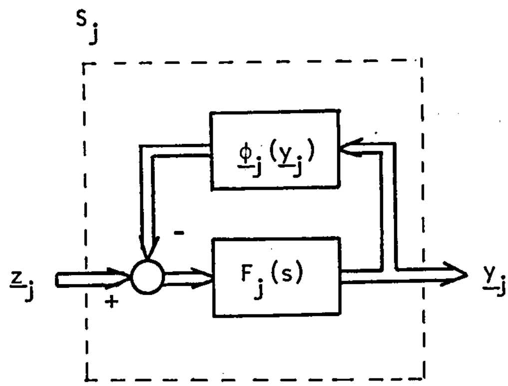

Fig. 5-2 (c)

Subsystem S_j of System IID

$0 \leqq \phi_k^{(j)}(y_j)y_k^{(j)} \leqq \phi_k^{(j)}[y_k^{(j)}]^2$

for $k=1, \cdots, s_j$

fact, by substituting $\tilde{\phi}_j(y_j) = \phi_j(y_j) - \zeta_j y_j$ we obtain

$$\begin{aligned}
\dot{\underline{x}}_j &= (A_j + \underline{b}_j\zeta_j\underline{c}_j^T)\underline{x}_j + \underline{b}_j\{-\tilde{\phi}_j(y_j) + z_j\} \\
y_j &= \underline{c}_j^T\underline{x}_j \qquad\qquad (5\text{-}31) \\
z_j &= g_j(y_1, \cdots, y_m, t) + \tilde{z}_j
\end{aligned}$$

and

$$0 \leqq \tilde{\phi}_j(y_j)y_j \leqq (\eta_j - \zeta_j)y_j^2$$

Remark 5-3

The block diagram of System IIC is illustrated in Fig. 5-2 (a) and each subsystem S_j is illustrated in Fig. 5-2 (b). $F_j(s)$ is an $s_j \times s_j$ transfer function given by

$$F_j(s) = C_j(sI - A_j)^{-1}B_j \qquad (5\text{-}32)$$

When $s_j = 1$ for all j, System IIC agrees with System IIA. Therefore, System IIC includes System IIA as a special case. From (5-18) and (5-19), the origin $\underline{x} = (\underline{x}_1^T, \cdots, \underline{x}_m^T)^T = \underline{0}$ is an equilibrium point for the null input $\underline{\tilde{z}} = (\underline{\tilde{z}}_1^T, \cdots, \underline{\tilde{z}}_m^T)^T = \underline{0}$.

Remark 5-4

System IID has the same structure as System IIC (Fig. 5-2 (a)) and each subsystem is illustrated in Fig. 5-2 (c). System IID is a special case of System IIC and Remark 5-3 holds in this case, too. When $s_j = 1$ for all j, System IID agrees with System IIB. Therefore, System IID includes System IIB as a special case. The condition (5-26) that the lower bounds of $\phi_k^{(j)}y_k^{(j)}$ are zero is not restrictive for the same reason as stated in Remark 5-2.

As pointed out in the above remarks, in each case, $\underline{x} = \underline{0}$ is an equilibrium point for the null input $\underline{\tilde{z}} = \underline{0}$. In the following sections, we will study the Lyapunov stability of the equilibrium $\underline{x} = \underline{0}$ for $\underline{\tilde{z}} \equiv \underline{0}$. First, we will obtain the stability criteria for System IIA and System IIB, then we will extend the results to System IIC and System IID, respectively.

Before we study stability, let us examine well-posedness of System IIA - System IID. A simple result can be obtained directly from

Corollary 2-1.

Theorem 5-1

Systems IIA - IID are well-posed in the sense of Definition 2-1.

[Proof] Consider System IIA. This system can be viewed as a single-loop system which consists of two blocks G and H where G is an m-input m-output system whose transfer matrix is $\mathrm{diag}(f_j(s))$ and H is also an m-input m-output whose characteristics is given by

$$(H\underline{\sigma})(t) = \begin{bmatrix} -\phi_1(\sigma_1, t) + g_1(\underline{\sigma}, t) \\ \vdots \\ -\phi_m(\sigma_m, t) + g_m(\underline{\sigma}, t) \end{bmatrix}$$

As the linear part $f_j(s)$ of each subsystem has no feed-through, the uniform instantaneous gain is zero by Theorem 2-1. Therefore the uniform instantaneous gain of G is zero and, so, System IIA has no loops consisting of subsystems which uniform instantaneous gains are not zero. By Corollary 2-1, System IIA is well-posed in the sense of Definition 2-1. This result clearly holds for the other three cases. [Q.E.D.]

We can consider more general situation that the linear part of each subsystem has a feed-through and, in this situation, we can obtain a quantitative condition for well-posedness. But, in this thesis, we do not consider this situation, because it becomes much more difficult to apply Theorem 3-2 to those systems whose subsystems have feed-through. We will explain the reason in more detail in the next section (see Remark 5-11).

Sec. 5.2. Circle Criteria Type Conditions for Composite Systems

In this section, we consider the Lyapunov stability conditions of System IIA and System IIC.

5.2.1. Frequency Domain Conditions of System IIA and System IIC

Theorem 5-2 (Stability condition of System IIA)

Consider System IIA given by (5-1)-(5-6). Assume each $(A_j, \underline{b}_j)$ is a controllable pair and let $f_j(s)$ be the transfer function given by (5-29). Then, the origin $\underline{x} = \underline{0}$ of this system is asymptotically stable in the large if there exists a positive number α_j for each j such that

$$h_j(s) = \frac{1 + (\eta_j + \alpha_j)f_j(s)}{1 + (\zeta_j - \alpha_j)f_j(s)} \tag{5-33}$$

is positive-real and if the mxm matrix $\hat{A} - \hat{B}$ is an M-matrix where

$$\hat{A} = \mathrm{diag}(\alpha_j), \quad \hat{B} = (\beta_{jk}) \quad \text{for } j, k = 1, \cdots, m \tag{5-34}$$

In the following, remarks are made to give more concrete ideas about the computational aspects of that procedure as well as to clarify theoretical backgrounds of the theorem.

Remark 5-5 (Positive realness)

A rational function h(s) of the complex variables with real coefficient is positive-real if h(s) has no poles in the open right half plane Re s > 0 and if the Nyquist locus of h(s) lies in the closed right half plane (see Lemma 4-3). The graphical interpretation of the requirement that our $h_j(s)$ given by (5-33) should be positive-real is well-known as the circle criterion. Namely, when $f_j(s)$ has no poles in the right half plane, the $h_j(s)$ of (5-32) is positive-real if the Nyquist locus of $f_j(s)$

(Case 1; when $\zeta_j - \alpha_j > 0$) does not encircle nor intersect the circle with

$$\text{center: } -\frac{1}{2}\left(\frac{1}{\zeta_j - \alpha_j} + \frac{1}{\eta_j + \alpha_j}\right) + 0i \tag{5-35}$$

$$\text{radius: } \frac{1}{2}\left|\frac{1}{\zeta_j - \alpha_j} - \frac{1}{\eta_j + \alpha_j}\right| \tag{5-36}$$

(Case 2; when $\zeta_j - \alpha_j = 0$) lies in the right of the line parallel to the imaginary axis and crossing the real axis at $-1/(\eta_j + \alpha_j)$, or

(Case 3; when $\zeta_j - \alpha_j < 0$) lies inside the circle with the center and radius given by (5-35) and (5-36).

When $f_j(s)$ has poles in the open right half plane, the encirclement condition must be altered. Refer to Chapters 3 and 7 of Narendra & Taylor (1973) for details.

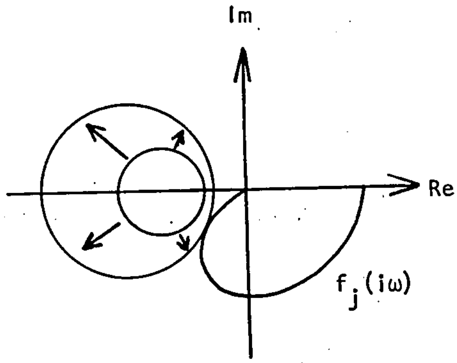

Fig. 5-3 Determination of α_j ($\alpha_j < \zeta_j$)

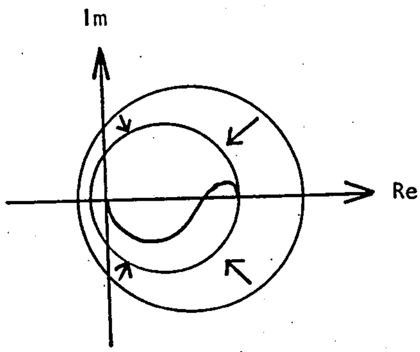

Fig. 5-4 Determination of α_j ($\alpha_j > \zeta_j$)

controllable and observable triplet. Let $F_j(s)$ be the transfer function given by (5-32). Then the origin $\underline{x} = \underline{0}$ of this system is asymptotically stable in the large if there exists a positive constant α_j and a positive-definite diagonal matrix $W_j = \mathrm{diag}(w_k^{(j)})$; $k=1, \cdots, s_j$ for each j such that

$$H_j(s) = W_j[I + F_j(s)(\zeta_j - \alpha_j I)]^{-1}[I + F_j(s)(\eta_j + \alpha_j I)]W_j^{-1} \qquad (5\text{-}37)$$

is strictly positive-real where

$$\eta_j = \mathrm{diag}(\eta_k^{(j)}), \quad \zeta_j = \mathrm{diag}(\zeta_k^{(j)}) \text{ for } k=1, \cdots, s_j \qquad (5\text{-}38)$$

and if the mxm matrix $\hat{A} - \hat{B}$ is an M-matrix where

$$\hat{A} = \mathrm{diag}(\alpha_j) \qquad (5\text{-}39)$$

$$\hat{B} = (\hat{\beta}_{jk})$$

$$\text{where } \hat{\beta}_{jk} = \sqrt{\lambda_{max}[W_k^{-1}\beta_{jk}^T \; W_j^2\beta_{jk}W_k^{-1}]} \qquad (5\text{-}41)$$

$$\text{for } j, k = 1, \cdots, m$$

When $s_j = 1$ for all j, this theorem agrees with Theorem 5-2. Namely, this theorem includes Theorem 5-2. When s_j is greater than 1, we cannot use such graphical technique as is stated in Remark 5-5 and we need to determine the value of the weight matrix W_j. In the following remarks, we give more concrete ideas about the computational aspects of the procedure.

Remark 5-9 (Determination of W_j)

$H_j(s)$ is strictly positive-real, if and only if there exists a positive constant ε such that

$$H_j(s-\varepsilon) = W_j[I + F_j(s-\varepsilon)(\zeta_j - \alpha_j I)]^{-1}[I + F_j(s-\varepsilon)(\eta_j + \alpha_j I)]W_j^{-1} \qquad (5\text{-}24)$$

is positive-real. Some properties of positive-real matrices are given in Lemma 4-3 - Lemma 4-6. In the above equality, if we set

$$W_j = \hat{W}, \; F_j(s-\varepsilon) = F(s), \; \eta_j + \alpha_j I = \eta, \; \zeta_j - \alpha_j I = \zeta \qquad (5\text{-}43)$$

we obtain (4-25) which is one of the alternative representation of the weighted multivariable circle criterion. Therefore, we can test the positive-reality of $H_j(s-\varepsilon)$ in the frequency domain by Theorem 4-2 and search a feasible W_j by the

Remark 5-6 (Determination of α_j)

Note that a matrix with non-positive off-diagonal elements is more likely to be an M-matrix if the diagonal elements are larger. This fact and the fact stated in Remark 5-5 justify the following way of determining α_j. Here, consider the case in which $f_j(s)$ has no poles in the open right half plane and $\zeta_j>0$. First draw the Nyquist locus of $f_j(s)$. Then, set $\alpha_j = 0$ and draw the circle with center (5-35) and radius (5-36). If the Nyquist locus is found to encircle or to intersect the above circle, Theorem 5-2 fails to assure asymptotic stability. If such is not the case, increase α_j until the circle with center (5-35) and radius (5-36) touches the Nyquist locus of $f_j(s)$. While α_j remains less than ζ_j, the circle grows as α_j increases (Fig. 5-3). If α_j exceeds ζ_j, the circle turns over and start shrinking (Fig. 5-4). In either case, the circle is sure to touch the Nyquist locus at a certain value of α_j and that value is exactly what we need in Theorem 5-2. The above operation can be easily carried out if a computer with a graphic terminal is available. The other cases can be treated in a parallel way.

Remark 5-7 (Relation of Theorem 5-2 to Corollary 4.2 of Araki (1976))

It would be evident from Remark 5-6 that the condition given in Theorem 5-2 is equivalent to the L_2-stability condition reported by Araki (1976) as Corollary 4.2 when no $f_j(s)$ has poles in the open right half plane.

Remark 5-8

From the proof of Theorem 5-2, this theorem assures the existence of a weighted sum of Lur'e type Lyapunov functions of subsystems (see 5.2.2). The relation between the frequency-domain condition and the existence of the above sort of Lyapunov functions was first suggested and partially proved in Araki (1978a) where the major interest lay in showing the parallelism between the Lyapunov stability analysis and the input-output stability analysis. Araki treated the special case of System IIA; i.e. $\zeta_j = 0$ and $\eta_j = \infty$ for all j, and the main tool was Theorem 3-1.

Theorem 5-3 (Stability condition of System IIC)

Consider System IIC given by (5-13)-(5-19). Assume each (A_j, B_j, C_j) is a

$$\tilde{h}(s) = \frac{1 + \tilde{\eta} f(s)}{1 + \tilde{\zeta} f(s)} \tag{5-45}$$

is strictly positive-real where

$$f(s) = \underline{c}^T (sI - A)^{-1} \underline{b}$$

[Note] If we assume that $A - \tilde{\zeta}\underline{b}\underline{c}^T$ is Hurwitz, the condition turns out to be necessary, too.

[Proof] By choosing the inversion formulas

$$[\tilde{\zeta}^{-1} + \underline{c}^T(sI - A)^{-1}\underline{b}]^{-1} = \tilde{\zeta} - \tilde{\zeta}\underline{c}^T(sI - A + \tilde{\zeta}\underline{b}\underline{c}^T)^{-1}\underline{b}\tilde{\zeta}$$

we obtain

$$\tilde{h}(s) = \underline{c}^T(sI - \tilde{A})^{-1}\underline{b}(\tilde{\eta} - \tilde{\zeta}) + 1$$

where

$$\tilde{A} = A - \tilde{\zeta}\underline{b}\underline{c}^T \tag{5-46}$$

Since $\tilde{h}(s)$ is strictly positive-real, $\tilde{A}$ has no poles in the closed right half plane. Therefore, by the Lefschitz form of the Yakubovich-Kalman lemma (p. 49 of Narendra & Taylor (1973)), there exist a positive-definite matrix P and a vector $\underline{\tilde{q}}$ which satisfy

$$\tilde{A}^T P + P\tilde{A} = -\underline{\tilde{q}}\underline{\tilde{q}}^T - \varepsilon Q \tag{5-47}$$

$$P\underline{b}(\tilde{\eta} - \tilde{\zeta}) - \underline{c} = \sqrt{2}\,\underline{\tilde{q}} \tag{5-48}$$

for a sufficiently small $\varepsilon \geq 0$. By substituting (5-46) into (5-47) and by using (5-48), we obtain (5-45). [Q.E.D.]

[Proof of Theorem 5-2] Let δ_0 be the minimum eigenvalue of the M-matrix $\hat{A} - \hat{B}$ and choose $0 < \delta < \delta_0$. Then $\hat{A} - \delta I - \hat{B}$ remains an M-matrix (Corollary to Theorem 6 of Araki (1972)). Note that

$$\delta < \delta_0 \leqq \alpha_j \qquad \text{for } j=1, \cdots, m \tag{5-49}$$

follows from the properties of M-matrices. Since $h_j(s)$ is positive-real and since (5-49) implies $\eta_j + \alpha_j - \delta > \zeta_j - \alpha_j + \delta$, we obtain that $\tilde{h}_j(s)$ given by

$$\tilde{h}_j(s) = \frac{1 + (\eta_j + \alpha_j - \delta) f_j(s)}{1 + (\zeta_j - \alpha_j + \delta) f_j(s)} \tag{5-50}$$

method proposed in Section 4-4.

Remark 5-10 (Determination of α_j)

Remark 5-9 and the M-matrix property mentioned in Remark 5-6 justify the following way of determining the value of α_j. First, set $\alpha_j = 0$ and determine W_j by the method of searching a feasible weight and next test the positive-reality of $H_j(s-\varepsilon)$ for $\alpha_j = 0$ by Theorem 4-2 (Here, we had better choose a positive number ε as small as possible). If $H_j(s-\varepsilon)$ with $\alpha_j = 0$ is found not to be positive-real for the values of W_j, there is no $\alpha_j>0$ such that $H_j(s-\varepsilon)$ is positive-real for the value of W_j. So, in this case, choose another value of W_j and go back to the second step, or stop applying Theorem 5-3. If such is not the case, increase α_j until the positive-reality test fails (During this process, we use the value of W_j which is determined in the first step). The last value of α_j is what we need in Theorem 5-3.

5.2.2. Proof of Theorem 5-2 and Theorem 5-3

To prove Theorem 5-2 we need the next lemma. This lemma is concerned with the well-known relation between the frequency domain condition and the existence of a positive-definite solution of a Riccati equation. It is essentially equivalent to Yakubovich-Kalman-Lefschitz lemma. The case $\zeta<0$ of this lemma was attributed to Rekasius & Rowland (1962) by Narendra & Taylor (1973). A Parallel lemma with $Q = 0$ is also found in Šiljak & Weissenberger (1970). Since any explicit statements or proofs which cover our necessity are not available in the reference, we state the lemma here with a brief proof.

Lemma 5-1

Let A be an $\tilde{n}x\tilde{n}$ matrix, $\underline{b}$ and $\underline{c}$ $\tilde{n}$-vectors, $\tilde{\zeta}$ and $\tilde{\eta}$ scalars which satisfy $\tilde{\eta} > \tilde{\zeta}$, and Q a positive-definite $\tilde{n}x\tilde{n}$ matrix. Assume $(A, \underline{b})$ is a completely controllable pair. Then, there exists a positive-definite matrix P which satisfies

$$PA + A^TP = -\frac{1}{2}\{P\underline{b}(\tilde{\eta} - \tilde{\zeta})^2\underline{b}^TP - P\underline{b}(\tilde{\eta} + \tilde{\zeta})\underline{c}^T - \underline{c}(\tilde{\eta} + \tilde{\zeta})\underline{b}^TP + \underline{c}\underline{c}^T\} - \varepsilon Q \tag{5-44}$$

for a sufficiently small $\varepsilon>0$ if is strictly positive-real. Therefore, by Lemma 5-1, there exist $\varepsilon_j > 0$, a positive-definite $n_j \times n_j$ matrix P_j and an n_j-vector $\underline{q}_j$ which satisfy

$$A_j^T P_j + P_j A_j = -\underline{q}_j \underline{q}_j^T + (\zeta_j - \alpha_j + \delta)(\eta_j + \alpha_j - \delta)\underline{c}_j \underline{c}_j^T - \varepsilon_j I$$
$$P_j \underline{b}_j - \frac{1}{2}(\zeta_j + \eta_j)\underline{c}_j = \underline{q}_j \tag{5-51}$$

Let us define $v_j(\underline{x}_j, t)$ by

$$v_j(\underline{x}_j, t) = \underline{x}_j^T P_j \underline{x}_j \tag{5-52}$$

and apply Theorem 3-2. For the CSD given by (3-1), the functions $\underline{\tilde{f}}_j$ and $\underline{\tilde{g}}_j$ are given by

$$\begin{aligned} \underline{\tilde{f}}_j(\underline{x}_j, t) &= A_j \underline{x}_j - \underline{b}_j \phi_j(y_j, t) \\ \underline{\tilde{g}}_j(\underline{x}_1, \cdots, \underline{x}_m, t) &= \underline{b}_j g_j(y_1, \cdots, y_m, t) \end{aligned} \tag{5-53}$$

and the isolated subsystem (3-6) turns out to be

$$\begin{aligned} \dot{\underline{x}}_j &= A_j \underline{x}_j - \underline{b}_j \phi_j(y_j, t) \\ y_j &= \underline{c}_j^T \underline{x}_j \end{aligned} \tag{5-54}$$

By (5-51), we obtain

$$\begin{aligned} \dot{v}_j|_{(5-54)} &= \underline{x}_j^T (A_j^T P_j + P_j A_j)\underline{x}_j - 2\phi_j(y_j, t)\underline{b}_j^T P_j \underline{x}_j \\ &= -\frac{1}{2}|\underline{x}_j^T P_j \underline{b}_j|^2 (\eta_j - \zeta_j + 2\alpha_j - 2\delta)^2 \\ &\quad - 2(\phi_j(y_j, t) - \frac{1}{2}(\eta_j + \zeta_j)\underline{c}_j^T \underline{x}_j)(\underline{x}_j^T P_j \underline{b}_j) \\ &\quad - \frac{1}{2}|\underline{c}_j^T \underline{x}_j|^2 - \varepsilon_j \underline{x}_j^T \underline{x}_j \end{aligned} \tag{5-55}$$

From the assumption (5-4) we obtain

$$|\phi_j(y_j, t) - \frac{1}{2}(\eta_j + \zeta_j)y_j| \leqq \frac{1}{2}(\eta_j - \zeta_j)|y_j| \tag{5-56}$$

Therefore, noting that $y_j = \underline{c}_j^T \underline{x}_j$, we obtain from (5-55)

$$\begin{aligned} \dot{v}_j|_{(5-54)} \leqq & -\frac{1}{2}(\eta_j - \zeta_j + 2\alpha_j - 2\delta_j)^2 [u_{1j}(\underline{x}_j)]^2 + (\eta_j - \zeta_j)u_{1j}(\underline{x}_j)u_{2j}(\underline{x}_j) \\ & - \frac{1}{2}[u_{2j}(\underline{x}_j)]^2 - u_{0j}(\underline{x}_j) \end{aligned} \tag{5-57}$$

where

$$u_{1j}(\underline{x}_j) = |\underline{x}_j^T P_j \underline{b}_j|, \quad u_{2j}(\underline{x}_j) = |\underline{c}_j^T \underline{x}_j|, \quad u_{0j}(\underline{x}_j) = \varepsilon_j \underline{x}_j^T \underline{x}_j \tag{5-58}$$

As for the interconnection, we obtain the next from (5-6) and (5-52).

$$|[\mathrm{grad}_j v_j]^T \underline{b}_j g_j(y_1, \cdots, y_m, t)| \leqq 2|\underline{x}_j^T P_j \underline{b}_j| \sum_{k=1}^{m} \beta_{jk}|y_k|$$

$$= 2u_{1j}(\underline{x}_j) \sum_{k=1}^{m} \beta_{jk} u_{2k}(\underline{x}_k) \qquad (5\text{-}59)$$

Thus, the requirements (a) and (b) of Theorem 3-2 are satisfied with the values of the constants κ_j, λ_j, μ_j and γ_{jk} being

$$\kappa_j = \frac{1}{2}(\eta_j - \zeta_j + 2\alpha_j - 2\delta_j)^2, \ \lambda_j = \frac{1}{2}(\eta_j - \zeta_j), \ \mu_j = \frac{1}{2}$$

$$\gamma_{jk} = \beta_{jk} \qquad \text{for } j, k = 1, \cdots, m \qquad (5\text{-}60)$$

So, the test matrix $K^{1/2}M^{1/2} - \tilde{\Gamma}$ in the requirement (c) of Theorem 3-2 are satisfied and, therefore, the origin of System IIA is asymptotically stable in the large. Thus we have proved Theorem 5-2. [Q.E.D.]

Remark 5-11

When $f_j(s)$ has a feed-through it is considerably difficult to decompose System IIA into the Lur'e type subsystems (5-54). In this case, System IIA is described by

$$\dot{\underline{x}}_j = A_j \underline{x}_j + \underline{b}_j\{-\phi_j(y_j, t) + z_j\} \qquad (5\text{-}1)$$

$$y_j = \underline{c}_j^T \underline{x}_j + d_j z_j \qquad (5\text{-}61)$$

$$z_j = g_j(y_1, \cdots, y_m, t) \qquad (5\text{-}3)$$

And ϕ_j becomes a function of not only $\underline{c}_j^T \underline{x}_j$ but also the other $\underline{c}_k^T \underline{x}_k$. Therefore, in order to obtain an isolated Lur'e type subsystem, we need at least to represent the nonlinear function ϕ_j as the sum of a function of $\underline{x}_j$ and a function of $\underline{x}_k$ ($k=1, \cdots, n$). By using (5-4) and (5-6), we can formally represent the system (5-1), (5-61), and (5-3) in the form of (5-53). But this representation was found to be unsuitable, because the stability condition obtained by applying Theorem 3-1 to this representation was much more conservative than Theorem 5-2. When the feed-through exists, the following decomposition seems to be appropriate.

$$\tilde{\underline{f}}_j(\underline{x}_j) = A_j \underline{x}_j$$

$$\tilde{\underline{g}}_j(\underline{x}_1, \cdots, \underline{x}_m, t) = \underline{b}_j\{-\phi_j(y_j, t) + g_j(y_1, \cdots, y_m, t)\}$$

The similar formulation can be found in McClamroch (1976). In the discrete time case, Araki & Kato (1981) used this decomposition and obtained a parallel result to Theorem 5-2 where each subsystem contains feed-through. In the next section, we use a Lur'e type Lyapunov function to analyze System IIB and System IID. In this case, this decomposition is not useful and the decomposition (5-53) seems to be essential.

We can prove Theorem 5-3 in a parallel way to the proof of Theorem 5-2. First, we give a lemma which is essentially equivalent to Anderson's Lemma (1967).

Lemma 5-2

Let A be an $\tilde{n}x\tilde{n}$ matrix, B an $\tilde{m}x\tilde{n}$ matrix, C an $\tilde{n}x\tilde{m}$ matrix, $\tilde{\zeta}$ and $\tilde{\eta}$ diagonal matrices which satisfy $\tilde{\eta} > \tilde{\zeta}$. Assume (A, B, C) is a completely controllable and completely observable triplet. Then there exists a positive-definite matrix P which satisfies

$$PA + A^TP = -\frac{1}{2}\{PB(\tilde{\eta} - \tilde{\zeta})^2W^{-2}B^TP - PB(\tilde{\eta} + \tilde{\zeta})C - C^T(\tilde{\eta} + \tilde{\zeta})B^TP + C^TW^2C\} - 2\varepsilon P \tag{5-62}$$

for a sufficiently small $\varepsilon > 0$ if

$$\tilde{H}(s) = W[I + F(s)\tilde{\zeta}]^{-1}[I + F(s)\tilde{\eta}]W^{-1} \tag{5-63}$$

is strictly positive-real where $W = \mathrm{diag}(w_j)$, $w_j > 0$ and

$$F(s) = C(sI - A)^{-1}B \tag{5-64}$$

[Proof] From (5-63), we obtain

$$\begin{aligned}\tilde{H}(s) &= W(I + C(sI - \tilde{A})^{-1}B(\tilde{\eta} - \tilde{\zeta}))W^{-1} \\ &= I + WC(sI - \tilde{A})^{-1}B(\tilde{\eta} - \tilde{\zeta})W^{-1}\end{aligned}$$

where

$$\tilde{A} = A - B\tilde{\zeta}C$$

Since $\tilde{H}(s)$ is strictly positive-real, $\tilde{H}(s-\varepsilon)$ is positive-real for sufficiently small $\varepsilon > 0$. Therefore, by Anderson's lemma (1967), there exists a positive-definite matrix P and matrices L and Q which satisfy

$$P(\tilde{A} + \varepsilon I) + (\tilde{A} + \varepsilon I)^TP = -LL^T \tag{5-65}$$

$$PB(\tilde{\eta} - \tilde{\zeta})W^{-1} = WC - LQ \tag{5-66}$$

$$Q^TQ = 2I \tag{5-67}$$

From the proof of Anderson's lemma, we can assume that Q is invertible. Then by substituting (5-64) into (5-65) and from (5-66) and (5-67), we obtain (5-62).

[Q.E.D.]

Note that from the above proof we need to choose the value of ε so that $\tilde{H}(s-\varepsilon)$ is positive-real.

[Proof of Theorem 5-3] Since $H_j(s)$ is strictly positive-real, by Lemma 5-2 there exists P_j which satisfies

$$\begin{aligned} P_jA_j + A_j^TP_j = &-\tfrac{1}{2}\{P_jB_j(\eta_j - \zeta_j)^2W^{-2}B_j^TP_j - P_jB_j(\eta_j + \zeta_j)C_j \\ &- C_j^T(\eta_j + \zeta_j)B_j^TP_j + C_j^TW_j^2C_j\} - 2\varepsilon_jP_j \end{aligned} \tag{5-68}$$

for a sufficiently small $\varepsilon > 0$. Let us define $v_j(\underline{x}_j, t)$ by

$$v_j(\underline{x}_j, t) = \underline{x}_j^TP_j\underline{x}_j \tag{5-69}$$

and apply Theorem 3-2. For the system given by (3-1), the functions $\tilde{\underline{f}}_j$ and $\tilde{\underline{g}}_j$ are given by

$$\tilde{\underline{f}}_j(\underline{x}_j, t) = A_j\underline{x}_j - B_j\underline{\phi}_j(\underline{y}_j, t)$$

$$\tilde{\underline{g}}_j(\underline{x}_1, \cdots, \underline{x}_m, t) = B_j\underline{g}_j(\underline{y}_1, \cdots, \underline{y}_m, t)$$

and the isolated subsystem turns out to be

$$\begin{aligned} \dot{\underline{x}}_j &= A_j\underline{x}_j - B_j\underline{\phi}_j(\underline{y}_j, t) \\ \underline{y}_j &= C_j\underline{x}_j \end{aligned} \tag{5-70}$$

By (5-68) we obtain

$$\begin{aligned} \dot{v}_j|_{(5\text{-}70)} &= \underline{x}_j^T(A_jP_j + P_jA_j)\underline{x}_j - 2\underline{\phi}_j^T(\underline{y}_j, t)B_j^TP_j\underline{x}_j \\ &= -\tfrac{1}{2}\{\underline{x}_j^TP_jB_j(\eta_j - \zeta_j + 2\alpha_jI)^2W_j^{-2}B_j^TP_j\underline{x}_j \\ &\quad - \underline{x}_j^TP_jB_j(\eta_j + \zeta_j)C_j\underline{x}_j - \underline{x}_j^TC_j^T(\eta_j + \zeta_j)B_j^TP_j\underline{x}_j + \underline{x}_j^TC_j^TW_j^2C_j\underline{x}_j\} \\ &\quad - 2\varepsilon_j\underline{x}_j^TP_j\underline{x}_j - 2\underline{\phi}_j^T(\underline{y}_j, t)B_j^TP_j\underline{x}_j \end{aligned} \tag{5-71}$$

From the assumption (5-17) we obtain

$$|\phi_k^{(j)}(y_k^{(j)},\ t) - \tfrac{1}{2}(\eta_k^{(j)} + \zeta_k^{(j)})y_k^{(j)}| \leqq |\eta_k^{(j)} - \zeta_k^{(j)}||y_k^{(j)}| \qquad (5\text{-}72)$$

$$\text{for } j=1,\ \cdots,\ m;\ k=1,\ \cdots,\ s_j$$

By (5-72)

$$|[\tfrac{1}{2}\,\underline{x}_j^T C_j^T(\eta_j + \zeta_j) - \underline{\phi}_j(\underline{y}_j,\ t)]B_j^T P_j \underline{x}_j|$$

$$\leqq |C_j\underline{x}_j|(\eta_j - \zeta_j)|B_j^T P_j \underline{x}_j| \qquad (5\text{-}73)$$

where the symbol $|\cdot|$ in the right-hand side of (5-73) is defined by (5-20).

By using (5-73), we obtain from (5-71)

$$\dot{v}_j|_{(5\text{-}70)} \leqq -\tilde{\kappa}_j[u_{1j}(\underline{x}_j)]^2 + 2\tilde{\lambda}_j u_{1j}(\underline{x}_j)u_{2j}(\underline{x}_j) - \tfrac{1}{2}[u_{2j}(\underline{x}_j)]^2 - u_{0j} \qquad (5\text{-}74)$$

where

$$\begin{aligned}
\tilde{\kappa}_j &= \tfrac{1}{2}\max_k(\eta_k^{(j)} - \zeta_k^{(j)} + 2\alpha_j)^2 \\
\tilde{\lambda}_j &= \tfrac{1}{2}\max_k(\eta_k^{(j)} - \zeta_k^{(j)}) \\
u_{1j}(\underline{x}_j) &= ||\underline{x}_j^T P_j B_j W_j^{-1}||,\quad u_{2j}(\underline{x}_j) = ||W_j C_j \underline{x}_j|| \\
u_{0j}(\underline{x}_j) &= 2\varepsilon_j \underline{x}_j^T P_j \underline{x}_j \qquad \text{for } j=1,\ \cdots,\ m;\ k=1,\ \cdots,\ s_j
\end{aligned} \qquad (5\text{-}75)$$

and $||\cdot||$ means Euclidean norm.

As for the interconnection, we obtain the next inequality from (5-19) and (5-69).

$$|[\mathrm{grad}_j v_j]^T B_j \underline{g}_j(\underline{y}_1,\ \cdots;\ \underline{y}_m,\ t)|$$

$$\leqq\ 2|\underline{x}_j^T P_j B_j W_j^{-1}|\ \sum_{k=1}^{m} W_j \beta_{jk} W_k^{-1}|W_k C_k \underline{x}_k|$$

by (5-41) and (5-75)

$$\leqq 2\ u_{1j}(\underline{x}_j)\ \sum_{k=1}^{m}\ \hat{\beta}_{jk} u_{2k}(\underline{x}_k) \qquad (5\text{-}76)$$

Thus the requirements (a), (b) of Theorem 3-2 are satisfied with the values of the constants κ_j, λ_j, μ_j and γ_{jk} being

$$\begin{aligned}
\kappa_j &= \tfrac{1}{2}\max_k(\eta_k^{(j)} - \zeta_k^{(j)} + 2\alpha_j)^2 \\
\lambda_j &= \tfrac{1}{2}\max_k(\eta_k^{(j)} - \zeta_k^{(j)}) \\
\mu_j &= \tfrac{1}{2} \qquad\qquad \text{for } j,\ \ell = 1,\ \cdots,\ m \\
\gamma_{j\ell} &= \hat{\beta}_{j\ell} \qquad\qquad k=1,\ \cdots,\ s_j
\end{aligned}$$

So, the test matrix $K^{1/2}M^{1/2} - \Gamma$ in the requirement (c) is equal to $\hat{A} - \hat{B}$. Since

$\hat{A} - \hat{B}$ is an M-matrix, all the requirements of Theorem 3-2 are satisfied. Therefore the origin of System IIC is asymptotically stable in the large.

[Q.E.D.]

5.2.3. Alternative Proof of Theorem 5-2

In Remark 5-7, we pointed out that Theorem 5-2 agrees with the L_2-stability condition. Usually, the input-output stability conditions tend to be sharper than the Lyapunov stability conditions for composite systems. Considering this fact and the agreement stated in Remark 5-7, we can expect that Theorem 5-2 is a fairly sharp condition among conditions which will be obtained by Theorem 3-2. But the proof of Theorem 5-2 does not give any information about the sharpness. To ascertain this expectation, we derive a stability condition of System IIA depending upon a principal idea of trying to make the stability condition as sharp as possible.

The next theorem is the main result in 5.2.3.

Theorem 5-4

Consider System IIA. Assume each $(A_j, \underline{b}_j)$ is a completely controllable pair and let $f_j(s)$ be given by (5-29). Then, the origin $\underline{x} = \underline{0}$ of System IIA is asymptotically stable, if there exist positive numbers $\tilde{\kappa}_j$ and $\tilde{\mu}_j$ for each j such that

$$h_j(s) = \frac{1 + \left(\frac{\eta_j + \zeta_j}{2} + \sqrt{\tilde{\kappa}_j\tilde{\mu}_j}\right) f_j(s)}{1 + \left(\frac{\eta_j + \zeta_j}{2} - \sqrt{\tilde{\kappa}_j\tilde{\mu}_j}\right) f_j(s)} \tag{5-77}$$

is positive-real and if the mxm matrix $\hat{A} - \hat{B}$ is an M-matrix where

$$\hat{A} = \mathrm{diag}\left(\sqrt{\tilde{\kappa}_j\tilde{\mu}_j} - \frac{\eta_j - \zeta_j}{2}\right), \quad \hat{B} = (\beta_{jk}) \tag{5-78}$$

If we set

$$\sqrt{\tilde{\kappa}_j\tilde{\mu}_j} = \alpha_j + \frac{\eta_j - \zeta_j}{2} \tag{5-79}$$

in Theorem 5-4, we immediately obtain Theorem 5-2. Namely Theorem 5-4 is equivalent to Theorem 5-2.

First, we roughly explain the difference between the proof of Theorem 5-2 and that of Theorem 5-3. In the proof of Theorem 5-2, first we gave the value of P_j as a solution of the Riccati equation (5-51) and next obtained the requirements (a) and (b) of Theorem 3-2 by setting u_{1j}, u_{2j} as (5-58). In the proof of Theorem 5-3, we assume that v_j, and u_{1j} and u_{2j} are given by (5-52) and (5-58), respectively, where the value of P_j has not been given yet. On this assumption, the requirement (b) of Theorem 3-2 is automatically satisfied and the main problem is how to obtain the requirement (a) of Theorem 3-2. We derive the condition of P_j so that the time-derivative of v_j along the solution of an isolated subsystem can be estimated as sharp as possible.

To prove Theorem 5-3, the next lemma is necessary.

Lemma 5-3

Let A be an $\tilde{n}x\tilde{n}$ matrix, $\underline{b}$ and $\underline{c}$ $\tilde{n}$-vectors, ζ and η scalars satisfying $\eta > 0$ and $\eta > \zeta$, and Q a positive-definite $\tilde{n}x\tilde{n}$ matrix. Assume $(A, \underline{b})$ is a completely controllable pair. Then, there exists a positive-definite matrix P which satisfies

$$PA + A^TP = -\tilde{\kappa}P\underline{b}\underline{b}^TP + \frac{\eta + \zeta}{2}P\underline{b}\underline{c}^T + \frac{\eta + \zeta}{2}\underline{c}\underline{b}^TP - \tilde{\mu}\underline{c}\underline{c}^T - \varepsilon Q \tag{5-80}$$

for a sufficiently small $\varepsilon > 0$ if

$$\tilde{h}(s) = \frac{1 + (\frac{\eta + \zeta}{2} + \sqrt{\tilde{\kappa}\tilde{\mu}})f(s)}{1 + (\frac{\eta + \zeta}{2} - \sqrt{\tilde{\kappa}\tilde{\mu}})f(s)} \tag{5-81}$$

is strictly positive-real where f(s) is given by

$$f(s) = \underline{c}^T(sI - A)^{-1}\underline{b} \tag{5-82}$$

[Proof of Lemma 5-3] All the assumptions of this theorem correspond to those of Lemma 5-1 by the relation

$$\tilde{\eta} = \frac{\eta + \zeta}{2} + \sqrt{\tilde{\kappa}\tilde{\mu}},\ \tilde{\zeta} = \frac{\eta + \zeta}{2} - \sqrt{\tilde{\kappa}\tilde{\mu}} \tag{5-83}$$

Therefore, if the assumptions of Lemma 5-3 is satisfied, then from (5-44), there exists a positive-definite solution $\hat{P}$ which satisfies

$$\hat{P}A + A^T\hat{P} = -\frac{1}{2}\{\hat{P}\underline{b}4\tilde{\kappa}\tilde{\mu}\underline{b}^T\hat{P} - \hat{P}\underline{b}(\eta + \zeta)\underline{c}^T + \underline{c}(\eta + \zeta)\underline{b}^T\hat{P} + \underline{c}\underline{c}^T\} - \hat{\varepsilon}Q \tag{5-84}$$

for a sufficiently small $\hat{\varepsilon} > 0$. By setting $\hat{P} = \frac{P}{2\tilde{\mu}}$ and $\hat{\varepsilon} = \frac{\varepsilon}{2}$ in (5-84), we obtain (5-80). [Q.E.D.]

[Proof of Theorem 5-4] Let us define $v_j(\underline{x}_j, t)$ by

$$v_j(\underline{x}_j, t) = \underline{x}_j^T P_j \underline{x}_j \tag{5-85}$$

where P_j is assumed to be a positive-definite matrix. Let us apply Theorem 3-2. For the system given by (3-1), the functions $\underline{\tilde{f}}_j$ and $\underline{\tilde{g}}_j$ are given by

$$\underline{\tilde{f}}_j(\underline{x}_j, t) = A_j\underline{x}_j - \underline{b}_j\phi_j(y_j, t)$$

$$\underline{\tilde{g}}_j(\underline{x}_1, \cdots, \underline{x}_m, t) = \underline{b}_j g_j(y_1, \cdots, y_m, t)$$

and the isolated subsystem (3-6) turns out to be

$$\dot{\underline{x}}_j = A_j\underline{x}_j - \underline{b}_j\phi_j(y_j, t), \; y_j = \underline{c}_j^T\underline{x}_j \tag{5-86}$$

By (5-12) and (5-85), we obtain the requirement (b) of Theorem 3-1

$$\begin{aligned}(\mathrm{grad}_j v_j)^T\underline{\tilde{g}}_j &\leqq 2|\underline{x}_j^T P_j \underline{b}_j| \sum_{k=1}^{m} \beta_{jk}|\underline{c}_k^T\underline{x}_k| \\ &= 2\, u_{1j}(\underline{x}_j) \sum_{k=1}^{m} \beta_{jk}u_{2k}(\underline{x}_k)\end{aligned} \tag{5-87}$$

where we put

$$u_{1j}(\underline{x}_j) = |\underline{x}_j^T P_j \underline{b}_j| \quad , \quad u_{2j}(\underline{x}_j) = |\underline{c}_j^T\underline{x}_j| \tag{5-88}$$

In the following, we examine the requirement (a) of Theorem 3-2. While we investigate this, we omit the suffix j which denotes the j-th subsystem (all the parameters except for the time t have this suffix). From the assumption (5-4), $\phi(y, t)$ satisfies

$$(\phi(\underline{c}^T\underline{x}, t) - \zeta\underline{c}^T\underline{x})(\eta\underline{c}^T\underline{x} - \phi(\underline{c}^T\underline{x}, t)) \geqq 0 \tag{5-89}$$

The sufficient condition to assure (3-16) is that the right-hand side of (3-16) is greater than or equal to $\dot{v}|_{(5-86)}$ for any nonlinear function ϕ satisfying (5-89)†; i.e.

$$\begin{aligned}q(\underline{x}) &= \min_{\substack{\phi \\ \text{subject to (5-89)}}}\{-\kappa u_1^2 + 2\lambda u_1 u_2 - \mu u_2^2 - u_0 - \dot{v}|_{(5-86)}\} \\ &\geqq 0\end{aligned} \tag{5-90}$$

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†) As the function ϕ is characterized only by (5-89), this method is considered to be the sharpest to obtain the estimate (3-16). This is the well-known technique used in the proof of the absolute stability problem.

Note that as each subsystem Lyapunov function v is defined by (5-85), the matrix P contained in $q(\underline{x})$ needs to be positive-definite. Let u_0 be given by $u_0 = \varepsilon\underline{x}^T Q\underline{x}$ and assume that

$$\lambda \geq 0 \tag{5-91}$$

then, from (5-88), $q(\underline{x})$ can be calculated as follows.

$$\begin{aligned} q(\underline{x}) &= \min_{\substack{\phi \\ \text{subject to (5-89)}}} \{- \kappa\underline{x}^T P\underline{b}\underline{b}^T P\underline{x} + 2\lambda|\underline{x}^T P\underline{b}||\underline{c}^T\underline{x}| - \mu\underline{x}^T\underline{c}\underline{c}^T\underline{x} - \varepsilon\underline{x}^T Q\underline{x} - \underline{x}^T(PA + A^T P)\underline{x} \\ &\qquad - 2\underline{x}^T P\underline{b}\phi(\underline{c}^T\underline{x}, t)\} \\ &\geq - \kappa\underline{x}^T P\underline{b}\underline{b}^T P\underline{x} + 2\tilde{\lambda}\underline{x}^T P\underline{b}\underline{c}^T\underline{x} - \mu\underline{x}^T\underline{c}\underline{c}^T\underline{x} - \varepsilon\underline{x}^T Q\underline{x} \\ &\qquad - \underline{x}^T(PA + A^T P)\underline{x} - 2 \min_{\substack{\phi \\ \text{subject to (5-89)}}} \{\underline{x}^T P\underline{b}\phi(\underline{c}^T\underline{x}, t)\} \end{aligned} \tag{5-92}$$

$$= \min \{\underline{x}^T R_1\underline{x},\ \underline{x}^T R_2\underline{x}\} \tag{5-93}$$

where R_1 and R_2 are constant matrices given by

$$R_1 = - \kappa P\underline{b}\underline{b}^T P + (\tilde{\lambda} + \zeta)P\underline{b}\underline{c}^T + (\tilde{\lambda} + \zeta)\underline{c}\underline{b}^T P - \mu\underline{c}\underline{c}^T - (PA + A^T P) - \varepsilon Q \tag{5-94}$$

$$R_2 = - \kappa P\underline{b}\underline{b}^T P + (\tilde{\lambda} + \eta)P\underline{b}\underline{c}^T + (\tilde{\lambda} + \eta)\underline{c}\underline{b}^T P - \mu\underline{c}\underline{c}^T - (PA + A^T P) - \varepsilon Q \tag{5-95}$$

and $\tilde{\lambda}$ is a real number satisfying

$$|\tilde{\lambda}| = \lambda \tag{5-96}$$

To derive (5-92), we used the next inequality which is evident from (5-91) and (5-96).

$$\lambda|\underline{x}^T P\underline{b}||\underline{c}^T\underline{x}| \geq \tilde{\lambda}\underline{x}^T P\underline{b}\underline{c}^T\underline{x} \tag{5-97}$$

From (5-93), we can conclude that (5-90) is satisfied if there exists a positive-definite matrix P for which R_1 and R_2 are simultaneously positive-semi-definite. In order to apply Lemma 5-3, we need to obtain one matrix equation which assures $R_1 \geq 0$ and $R_2 \geq 0$ simultaneously. Assume that there exists a positive-definite solution P and a constant number a such that

$$- \kappa P\underline{b}\underline{b}^T P + aP\underline{b}\underline{c}^T + a\underline{c}\underline{b}^T P - \mu\underline{c}\underline{c}^T - (PA + A^T P) - \varepsilon Q = 0 \tag{5-98}$$

If we put $\tilde{\lambda} = a - \zeta$ in (5-94), we obtain $\underline{x}^T R_1\underline{x} = 0$ and $\underline{x}^T R_2\underline{x} \geq 0$ for $\underline{x} \in \{\underline{x} \mid \underline{x}^T P\underline{b}\underline{c}^T\underline{x} \geq 0\}$, and if we put $\tilde{\lambda} = a - \eta$ in (5-95), we obtain $\underline{x}^T R_2\underline{x} = 0$ and $\underline{x}^T R_1\underline{x} \geq 0$ for $\underline{x} \in \{\underline{x} \mid \underline{x}^T P\underline{b}\underline{c}^T\underline{x} \leq 0\}$. Therefore, in each case, $\dot{v}|_{(5\text{-}86)}$ can be bounded as

$$\dot{v}|_{(5\text{-}86)} \leq -\kappa|\underline{x}^T P\underline{b}|^2 + 2(a-\zeta)\underline{x}^T P\underline{b}\underline{c}^T\underline{x} - \mu|\underline{x}^T\underline{c}|^2 - \varepsilon\underline{x}^T Q\underline{x}$$

$$\text{for } \underline{x} \in \{\underline{x} \mid \underline{x}^T P\underline{b}\underline{c}^T\underline{x} \geq 0\} \qquad (5\text{-}99)$$

$$\dot{v}|_{(5\text{-}86)} \leq -\kappa|\underline{x}^T P\underline{b}|^2 + 2(a-\eta)\underline{x}^T P\underline{b}\underline{c}^T\underline{x} - \mu|\underline{x}^T\underline{c}|^2 - \varepsilon\underline{x}^T Q\underline{x}$$

$$\text{for } \underline{x} \in \{\underline{x} \mid \underline{x}^T P\underline{b}\underline{c}^T\underline{x} \leq 0\} \qquad (5\text{-}100)$$

From (5-99) and (5-100)

$$\dot{v}|_{(5\text{-}86)} \leq -\kappa u_1^2 + 2\max\{|a-\zeta|,\ |a-\eta|\}\, u_1 u_2 - \mu u_2^2 - u_0$$

$$\text{for all } \underline{x} \qquad (5\text{-}101)$$

As the coefficient $\max\{|a-\zeta|, |a-\eta|\}$ takes the minimum value $(\eta-\zeta)/2$ for $a = (\eta+\zeta)/2$, we choose $a = (\eta+\zeta)/2$ to obtain a sharper estimate of $\dot{v}|_{(5\text{-}86)}$. Thus, as for the requirement (a) of Theorem 3-2, we obtain the next result;

$\dot{v}_j|_{(5\text{-}86)}$ can be bounded as

$$\dot{v}|_{(5\text{-}86)} \leq -\kappa_j u_{1j}^2 + 2\,\frac{\eta_j-\zeta_j}{2}\,u_{1j}u_{2j} - \mu_j u_{2j}^2 - \varepsilon_j \underline{x}_j^T Q_j \underline{x}_j$$

$$\text{for } j=1, \cdots, m \qquad (5\text{-}102)$$

if there exists a positive-definite matrix P_j satisfying

$$-\kappa_j P_j \underline{b}_j \underline{b}_j^T P_j + \frac{\eta_j+\zeta_j}{2} P_j \underline{b}_j \underline{c}_j^T + \frac{\eta_j+\zeta_j}{2} \underline{c}_j \underline{b}_j^T P_j - \mu_j \underline{c}_j \underline{c}_j^T - (P_j A_j + A_j^T P_j)$$

$$-\varepsilon_j Q_j = 0 \qquad (5\text{-}103)$$

$$\text{for } j=1, \cdots, m$$

Lastly, we examine the requirement (c) of Theorem 3-2. Let δ_0 be the minimum eigenvalue of the M-matrix and choose $0 < \delta < \delta_0$. Then, $\hat{A} - \hat{B} - \delta I$ remains an M-matrix. Since $h_j(s)$ given by (5-81) is positive-real, we obtain that $\tilde{h}_j(s)$ given by

$$\tilde{h}_j(s) = \frac{1 + \left(\frac{\eta_j+\zeta_j}{2} + \sqrt{\tilde{\kappa}_j\tilde{\mu}_j} - \delta\right) f_j(s)}{1 + \left(\frac{\eta_j+\zeta_j}{2} - \sqrt{\tilde{\kappa}_j\tilde{\mu}_j} - \delta\right) f_j(s)}$$

is strictly positive-real. From Lemma 5-3, there exists a positive-definite matrix P_j which satisfies (5-103) with $\sqrt{\kappa_j\mu_j} = \sqrt{\tilde{\kappa}_j\tilde{\mu}_j} - \delta$. Thus, the requirements (a) and (b) of Theorem 3-2 are satisfied with the values of the constants κ_j, λ_j, μ_j and γ_{jk} given by

$$\sqrt{\kappa_j \mu_j} = \sqrt{\tilde{\kappa}_j \tilde{\mu}_j} - \delta, \; \lambda_j = \frac{1}{2}(\eta_j - \zeta_j), \; \gamma_{jk} = \beta_{jk}$$

So, the test matrix $K^{1/2}M^{1/2} - \tilde{\Gamma}$ in the requirement (c) is equal to $\hat{A} - \hat{B} - \delta I$. Since $\hat{A} - \hat{B} - \delta I$ is an M-matrix from the choice of δ, we can conclude that all the requirements of Theorem 3-2 are satisfied and, therefore, the origin of System IIA is asymptotically stable. Thus we have proved Theorem 5-4. [Q.E.D.]

Theorem 5-3 can be proved in the same way as Theorem 5-4.

Sec. 5.3. Popov Criteria Type Condition for Composite Systems

In this section, we will consider the stability conditions of System IIB and System IID.

5.3.1. Frequency Domain Conditions of System IIB and System IID

Theorem 5-5 (Stability condition of System IIB)

Consider System IIB defined by (5-7)-(5-12). Assume that each $(A_j, \underline{b}_j)$ is a completely controllable pair, all the eigenvalues of A_j have negative real parts, and for the case A_j has some repeated eigenvalues, additionally that $(\underline{c}_j + \theta_j \underline{c}_j A_j, A_j)$ is a completely observable pair. Let $f_j(s)$ be the transfer function given by (5-29). Then the origin $\underline{x} = \underline{0}$ of System IIB is stable if there exist positive numbers κ_j and μ_j and a constant θ_j for each j such that

$$\frac{1}{\eta_j} + \mathrm{Re}\{(1 + \theta_j s) f_j(s)\} - \frac{\kappa_j}{2} |(1 + \theta_j s) f_j(s)|^2 - \mu_j \{\frac{\kappa_j}{\eta_j} + \frac{1}{2}\} |f_j(s)|^2 \geq 0 \tag{5-104}$$

where $s = i\omega$; $0 \leq \omega < \infty$, for $j=1, \cdots, m$

and

$$q_j(\kappa_j, \theta_j) = 1 - \frac{\eta_j}{2}\{\kappa_j \theta_j^2 (\underline{c}_j^T \underline{b}_j)^2 - 2\theta_j \underline{c}_j^T \underline{b}_j\} > 0 \;, \tag{5-105}$$

and if the mxm matrix $\hat{A} - \hat{B}$ is an M-matrix where

$$\hat{A} = \mathrm{diag}(\sqrt{\kappa_j \mu_j}), \; \hat{B} = (\beta_{jk}) \tag{5-106}$$

Additionally, if (5-104) is satisfied with $s = i\omega - \varepsilon$ for sufficiently small $\varepsilon > 0$, then System IIB is asymptotically stable in the large.

In the following, remarks are made to give more concrete ideas about the computational aspects of that procedure.

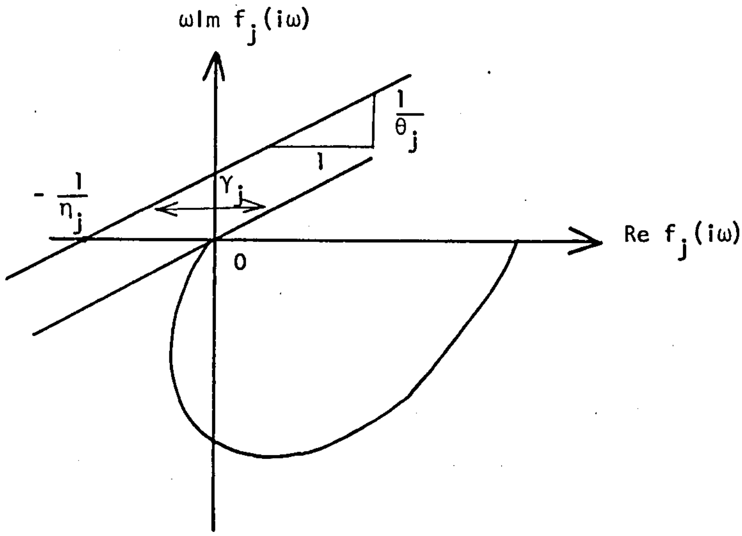

Fig. 5-5   Popov Locus and Popov Line

Remark 5-12

If we set $\kappa_j = \mu_j = 0$ in (5-104), we obtain

$$\gamma_j = \frac{1}{\eta_j} + \mathrm{Re}\{(1 + \theta_j s) f_j(s)\} > 0 \qquad (5\text{-}107)$$

This is the Popov criterion for a single-input single-output system and can be interpreted graphically; The inequality (5-107) is satisfied, if Popov locus of $f_j(s)$ lies to the right of the Popov line which goes through the point $(-1/\eta_j, 0)$ at a slope of $1/\theta_j$ (Fig. 5-5). The third and the fourth terms (5-104) are considered to be the stability margine required of each subsystem so that the whole system is guaranteed to be stable. As the value of γ_j defined by (5-107) becomes larger, (5-104) is expected to be satisfied for larger values of κ_j and μ_j.

Remark 5-13

When $sf_j(s)$ is strictly proper (i.e., $\underline{c}_j^T \underline{b}_j = 0$), (5-105) is obviously satisfied. As η_j and κ_j are positive, the necessary condition of (5-105) is

$$1 + \eta_j \theta_j \underline{c}_j^T \underline{b}_j > 0$$

This inequality can be found also in the Popov criterion.

Remark 5-14 (Determination of $\sqrt{\kappa_j \mu_j}$)

The M-matrix property stated in Remark 5-5 and the fact stated in Remark 5-12 and 5-13 justify the following way of determining the value of $\sqrt{\kappa_j \mu_j}$.

First, set $\kappa_j = 0$ and $\mu_j = 0$, and test (5-104) graphically (Remark 5-12). In this step, we determine θ_j so that γ_j in (5-107) becomes larger (Fig. 5-5). If, for some j, there exists no θ_j that satisfies (5-107), Theorem 5-5 fails to assure stability. If such is not the case, increase $\alpha_j = \sqrt{\kappa_j \mu_j}$ until the pair (κ_j, μ_j) does not satisfy (5-104) and (5-105). The last value is what we need in Theorem 5-5. To test the existence of (κ_j, μ_j), we may check, by point by point method, the existence of κ_j which satisfies (5-104) with $\mu_j = \alpha_j^2/\kappa_j$ in the interval

$$0 < \kappa_j < (1 + \eta_j \theta_j \underline{c}_j^T \underline{b}_j)/\theta_j^2 (\underline{c}_j^T \underline{b}_j)^2 \qquad (\text{if } \theta_j \underline{c}_j^T \underline{b}_j \neq 0)$$

$$0 < \kappa_j \qquad (\text{if } \theta_j \underline{c}_j^T \underline{b}_j = 0)$$

Theorem 5-6 (Stability condition of System IID)

Consider System IID given by (5-22)-(5-28). Assume that each (A_j, B_j) is a completely controllable pair, all the eigenvalues of A_j have negative real parts, and for the case A_j has some repeated eigenvalues, additionally that $(C_j + \theta_j C_j A_j, A_j)$ is a completely observable pair. Let $F_j(s)$ be the transfer function given by (5-32). Then, the origin $\underline{x} = \underline{0}$ of system IID is stable if the following two conditions are satisfied.

(a) For each j, there exist positive-definite diagonal matrices $W_j = \mathrm{diag}(w_k^{(j)})$ and $\Theta_j = \mathrm{diag}(\theta_k^{(j)})$ for $k=1, \cdots, s_j$ and positive numbers κ_j and μ_j such that

$$\{I - \hat{D}_j\hat{H}_j^{-1}[-\Theta_j\tilde{C}_j\tilde{B}_j + (I + s\Theta_j)\tilde{F}_j(s)]\}^*\{\kappa_j I + \hat{D}_j\hat{H}_j^{-1}\hat{D}_j\}^{-1}$$
$$\times\{I - \hat{D}_j\hat{H}_j^{-1}[-\Theta_j\tilde{C}_j\tilde{B}_j + (I + s\Theta_j)\tilde{F}_j(s)]\} - \mu_j\tilde{F}_j^*(s)\tilde{F}_j(s)$$
$$- [-\Theta_j\tilde{C}_j\tilde{B}_j + (I + s\Theta_j)\tilde{F}_j(s)]^*\hat{H}_j^{-1}[-\Theta_j\tilde{C}_j\tilde{B}_j + (I + s\Theta_j)\tilde{F}_j(s)] \geqq 0$$
$$\text{for } s = i\omega \;;\quad 0 \leqq \omega < \infty \tag{5-108}$$

where

$$\tilde{F}_j(s) = W_jF_j(s)W_j^{-1},\ \tilde{B}_j = B_jW_j^{-1},\ \tilde{C}_j = W_jC_j$$
$$\hat{D}_j = \kappa_j\tilde{B}_j^T\tilde{C}_j^T\Theta_j - I \tag{5-109}$$
$$\hat{H}_j = 2\eta_j^{-1} - \kappa_j\Theta_j\tilde{C}_j\tilde{B}_j\tilde{B}_j^T\tilde{C}_j^T\Theta_j + \Theta_j\tilde{C}_j\tilde{B}_j + \tilde{B}_j^T\tilde{C}_j^T\Theta_j$$

and

$$\hat{H}_j > 0 \text{ and } \det \hat{D}_j \neq 0 \tag{5-110}$$

(b) if the mxm matrix $\hat{A} - \hat{B}$ is an M-matrix where

$$\hat{A} \doteq \mathrm{diag}(\sqrt{\kappa_j\mu_j})$$
$$\hat{B} = (\hat{\beta}_{jk}), \qquad \hat{\beta}_{jk} = \sqrt{\lambda_{max}\,[W_k^{-1}\beta_{jk}^TW_j^2\beta_{jk}W_k^{-1}]} \tag{5-111}$$

Additionally, if (5-108) is satisfied with $s = i\omega - \varepsilon$ for sufficiently small $\varepsilon > 0$, then System IID is asymptotically stable in the large.

In the following remarks, we give more concrete ideas about the computation aspects of the procedure.

Remark 5-15 (Determination of Θ_j and W_j)

As (5-18) includes $2s_j$ arbitrary parameters $\Theta_j = \mathrm{diag}(\theta_k^{(j)})$ and $W_j = \mathrm{diag}(w_k^{(j)})$ (for $k=1, \cdots, s_j$), we need to determine these values before testing (5-108). If we set $\kappa_j = \mu_j = 0$ in (5-108), we obtain, after some calculation,

$$W_j^{-1}\{[\eta_j^{-1}W_j^2 + (W_j^2 + sW_j^2\Theta_j)F_j(s)]^* + [\eta_j^{-1}W_j^2 + (W_j^2 + sW_j^2\Theta_j)F_j(s)]\}W_j^{-1} \geqq 0 \tag{5-112}$$

where $s = i\omega$; $\quad 0 \leqq \omega < \infty$

This is Popov criterion for multi-input multi-output systems and also a necessary condition of (5-108). Let us determine the values of W_j and Θ_j by using (5-112) instead of (5-108). First, we consider the determination of Θ_j. The next properties are useful.

(i) In the same way as the proof of Theorem 4-3 (ii), we can obtain the next sufficient condition of (5-112); *there exists a positive-definite diagonal matrix W_j which satisfies (5-112), if $\Gamma_j = (\gamma_{k\ell}^{(j)})$ is an M-matrix where*

$$\begin{aligned} \gamma_{kk}^{(j)} &= \min_{\omega} \{(\eta_k^{(j)})^{-1} + Re(1 + i\omega\theta_k^{(j)})[F_j(i\omega)]_{kk}\} \\ \gamma_{k\ell}^{(j)} &= -\max_{\omega}|(1 + i\omega\theta_k^{(j)})[F_j(i\omega)]_{k\ell}| \qquad (k\neq\ell) \\ &\qquad\qquad for\ k,\ \ell = 1, \cdots, s_j \end{aligned} \tag{5-113}$$

From an M-matrix property, Γ_j is more likely to be an M-matrix, if γ_{kk} becomes larger and $-\gamma_{k\ell}$ $(k\neq\ell)$ becomes smaller.

(ii) As the diagonal elements of a positive-semi-definite Hermitian matrix are non-negative, the necessary condition of (5-112) is

$$(\eta_k^{(j)})^{-1} + \mathrm{Re}\{(1 + i\omega\theta_k^{(j)})[F_j(i\omega)]_{kk}\} \geqq 0 \tag{5-114}$$

$$\text{for all } \omega \text{ and } k=1, \cdots, s_j$$

This condition is also a necessary condition of (5-113), and can be easily tested graphically (Remark 5-10).

From these facts, one simple method is to obtain the values of $\theta_k^{(j)}$ graphically which makes $\gamma_{kk}^{(j)}$ as large as possible. As the off-diagonal elements of Γ_j become larger if the values of $\theta_k^{(j)}$ become larger, we cannot choose so large values of $\theta_k^{(j)}$. In Chapter 7, for a simple example, we will obtain the best $\theta_k^{(j)}$ so that Γ_j is more likely to be an M-matrix (Unfortunately we cannot examine (5-112) directly).

Next, let us consider how to determine the value of W_j on condition that the value of Θ_j has already been given. In this case, (5-112) can be regarded as (4-13) and we can determine W_j by the method of searching a feasible weight given in Sec. 4.4.

Remark 5-16 (Determination of $\sqrt{\kappa_j \mu_j}$)

In the first step, set $\kappa_j = \mu_j = 0$ and determine the values of Θ_j and W_j for each j by the method stated in Remark 5-15. If, for some (j, k), there exists no $\theta_k^{(j)}$ which satisfies (5-114), then Theorem 5-6 fails to assure stability. In the second step, test (5-108) for each j with $\kappa_j = 0$ and $\mu_j = 0$. If the test fails, there exist no positive numbers κ_j and μ_j for those values of Θ_j and W_j. In this case, choose another values of Θ_j and W_j and go back to the second step, or stop this procedure. If such is not the case, increase $\alpha_j = \sqrt{\kappa_j \mu_j}$ until the pair (κ_j, μ_j) which satisfies (5-108) and (5-110) does not exist. The last value α_j is what we wanted in Theorem 5-6. To test the existence of (κ_j, μ_j), we may check, by point by point method, the existence of κ_j which satisfies (5-108) with $\mu_j = \alpha_j^2/\kappa_j$ in the interval

$$0 < \kappa_j < \sqrt{\lambda_{max}[(\Theta_j \tilde{C}_j \tilde{B}_j \tilde{B}_j^T \tilde{C}_j^T \Theta_j)(2\eta_j^{-1} + \Theta_j \tilde{C}_j \tilde{B}_j + \tilde{B}_j^T \tilde{C}_j^T \Theta_j)^{-1}} \qquad (\text{if } \Theta_j \tilde{C}_j \tilde{B}_j \neq 0)$$

$$0 < \kappa_j \qquad (\text{if } \Theta_j \tilde{C}_j \tilde{B}_j = 0)$$

and

$$\det(\kappa_j \tilde{B}_j^T \tilde{C}_j^T \Theta_j - I) \neq 0$$

Remark 5-17

If each subsystem is single-input single-output, i.e., $s_j = 1$ (j=1, ···, m) (5-108) agrees with (5-104) though the former seems to be more complicated than the latter. Namely, Theorem 5-6 includes Theorem 5-5. Especially, when $\lim_{s \to 0} sF_j(s) = 0$ (namely $\tilde{C}_j \tilde{B}_j = 0$), the condition (5-108) becomes much simpler;

$$\begin{aligned}[I + \tfrac{\eta_j}{2}(I + s\Theta_j)\tilde{F}_j(s)]^* [\kappa_j I + 2\eta_j^{-1}]^{-1} [I + \tfrac{\eta_j}{2}(I + s\Theta_j)\tilde{F}_j(s)] \\ - \mu_j \tilde{F}_j^*(s)\tilde{F}_j(s) - [(I + s\Theta_j)\tilde{F}_j(s)]^* 2\eta_j^{-1} [(I + s\Theta_j)\tilde{F}_j(s)] \geqq 0 \\ \text{for } s = i\omega;\ 0 \leq \omega < \infty \end{aligned} \qquad (5\text{-}115)$$

5.3.2. Proof of Theorem 5-5 and Theorem 5-6

To prove Theorem 5-5 and Theorem 5-6, we need the next lemma.

Lemma 5-4

Assueme that $(\hat{\hat{A}}, \hat{\hat{B}})$ is controllable, all the eigenvalues of $\hat{\hat{A}}$ have negative real parts, $\hat{\hat{M}} \geqq 0$ and $\hat{\hat{K}}$ is invertible, and for the case where $\hat{\hat{A}}$ has some repeated eigenvalues, additionally that $(\hat{\hat{C}}, \hat{\hat{A}})$ is observable. Then the following properties are equivalent.

(a)
$$\hat{\hat{K}} + \hat{\hat{C}}(-i\omega I - \hat{\hat{A}})^{-1}\hat{\hat{B}} + \hat{\hat{B}}^T(i\omega I - \hat{\hat{A}}^T)^{-1}\hat{\hat{C}}^T - \hat{\hat{B}}^T(-i\omega I - \hat{\hat{A}}^T)^{-1}\hat{\hat{M}}(i\omega I - \hat{\hat{A}})^{-1}\hat{\hat{B}} \geqq 0 \text{ for all } \omega \tag{5-116}$$

(b) there exists a positive-definite matrix P which satisfies

$$\hat{\hat{M}} + P\hat{\hat{A}} + \hat{\hat{A}}^T P + (P\hat{\hat{B}} - \hat{\hat{C}}^T)\hat{\hat{K}}^{-1}(P\hat{\hat{B}} - \hat{\hat{C}}^T)^T = 0 \tag{5-117}$$

This lemma is obtained from the result of Hattori & Kobayashi (1980) immediately. Hattori and Kobayashi obtained their result utilizing the positive systems theory of Popov. This lemma deeply relates to the Anderson's lemma which was used in the proof of Theorem 5-3.

[Proof of Theorem 5-5] Let us define $v_j(\underline{x}_j, t)$ by

$$v_j(\underline{x}_j, t) = \underline{x}_j^T P_j \underline{x}_j + 2\theta_j \int_0^{\underline{c}_j^T \underline{x}_j} \phi_j(\sigma)d\sigma \tag{5-118}$$

where P_j is assumed to be a positive-definite matrix. Let us apply Theorem 3-2. For the system given by (3-1), the functions $\underline{\tilde{f}}_j$ and $\underline{\tilde{g}}_j$ are given by

$$\underline{\tilde{f}}_j(\underline{x}_j, t) = A_j\underline{x}_j - \underline{b}_j\phi_j(y_j)$$
$$\underline{\tilde{g}}_j(\underline{x}_1, \cdots, \underline{x}_m, t) = \underline{b}_j g_j(y_1, \cdots, y_m, t)$$

and the isolated subsystem (3-6) turns out to be

$$\dot{\underline{x}}_j = A_j\underline{x}_j - \underline{b}_j\phi_j(y_j), \quad y_j = \underline{c}_j^T\underline{x}_j \tag{5-119}$$

By (5-12) and (5-118), we obtain the requirement (b) of Theorem 3-1.

$$(\text{grad}_j v_j)^T\underline{\tilde{g}}_j = (2\underline{x}_j^T P_j + 2\theta_j\underline{c}_j^T\phi_j)\underline{b}_j g_j$$
$$\leqq 2|\underline{x}_j^T P_j \underline{b}_j + \theta_j\underline{c}_j^T\phi_j\underline{b}_j| \sum_{k=1}^{m} \beta_{jk}|\underline{c}_k^T\underline{x}_k|$$

$$= 2u_{1j}(\underline{x}_j) \sum_{k=1}^{m} \beta_{jk} u_{2k}(\underline{x}_k) \tag{5-120}$$

where we define u_{1j} and u_{2j} as

$$u_{1j}(\underline{x}_j) = |\underline{x}_j^T P_j \underline{b}_j + \theta_j \underline{c}_j^T \phi_j \underline{b}_j|, \quad u_{2j}(\underline{x}_j) = |\underline{c}_j^T \underline{x}_j| \tag{5-121}$$

Next, let us examine the requirement (a) of Theorem 3-2. We will estimate $\dot{v}_j|_{(5\text{-}119)}$ in the form of (3-16) with $\lambda_j = 0$ and $u_{0j} = 0$; i.e.

$$\dot{v}_j|_{(5\text{-}119)} \leqq -\kappa_j \{\underline{x}_j^T P_j \underline{b}_j + \theta_j \underline{c}_j^T \underline{b}_j \phi_j\}^2 - \mu_j \{\underline{x}_j^T \underline{c}_j\}^2 \tag{5-122}$$

While we investigate this requirement (a), we omit the suffix j which denotes the j-th subsystem (all the parameters except for time t needs this suffix).

From (5-118) and (5-119) we obtain

$$\begin{aligned} \dot{v}|_{(5\text{-}119)} &= \underline{x}^T(PA + A^T P)\underline{x} - 2\underline{x}^T P \underline{b} \phi(\underline{c}^T \underline{x}) \\ &\quad + 2\theta\phi(\underline{c}^T \underline{x})\underline{c}^T A\underline{x} - 2\theta\phi(\underline{c}^T \underline{x})\underline{c}^T \underline{b}\phi(\underline{c}^T \underline{x}) \end{aligned} \tag{5-123}$$

Since, by the assumption (5-10), $\phi(y)$ satisfies

$$\phi(\underline{c}^T \underline{x})(\eta \underline{c}^T \underline{x} - \phi(\underline{c}^T \underline{x})) \geqq 0 \tag{5-124}$$

(5-122) is satisfied if the right-hand side of (5-122) is greater than or equal to (5-123) for any nonlinear function ϕ which satisfies (5-124); i.e.,

$$\min_{\phi} \; (\hat{g}\phi^2 + \hat{e}\phi + \hat{q}) \geqq 0 \tag{5-125}$$

subject to (5-124)

where $\hat{g}$, $\hat{e}$ and $\hat{q}$ are scalars given by

$$\begin{aligned} \hat{g} &= -\kappa(\theta \underline{c}^T \underline{b})^2 + 2\theta \underline{c}^T \underline{b} \\ \hat{e} &= -2\kappa \underline{x}^T P \underline{b} \theta \underline{c}^T \underline{b} + 2\underline{x}^T P \underline{b} - 2\theta \underline{c}^T A \underline{x} \\ \hat{q} &= -\kappa(\underline{x}^T P \underline{b})^2 - \mu(\underline{x}^T \underline{c})^2 - \underline{x}^T(PA + A^T P)\underline{x} \end{aligned} \tag{5-126}$$

Assume that $\hat{g} + 2/\eta > 0$. Then, the next inequality is obviously satisfied.

$$\begin{aligned} \min_{\substack{\phi \\ \text{subject to (5-124)}}} (\hat{g}\phi^2 + \hat{e}\phi + \hat{q}) &\geqq \min_{\phi} \{\hat{g}\phi^2 + \hat{e}\phi + \hat{q} - 2\phi(\underline{c}^T \underline{x} - \phi/\eta)\} \\ &= \frac{4(\hat{g} + 2/\eta)^2 \hat{q} - (\hat{e} - 2\underline{c}^T \underline{x})^2}{4(\hat{g} + 2/\eta)^2} \end{aligned}$$

$$= - \frac{1}{\hat{g} + 2/\eta} \underline{x}^T \hat{R} \underline{x} \tag{5-127}$$

where $\hat{R}$ is a constant matrix given by

$$\hat{R} = PA + A^T P + \mu \underline{c}\underline{c}^T + \kappa P \underline{b}\underline{b}^T P + \frac{(\kappa\theta \underline{c}^T \underline{b} - 1)^2}{2/\eta - \kappa\theta^2 (\underline{c}^T \underline{b})^2 + 2\theta \underline{c}^T \underline{b}} P \underline{b}\underline{b}^T P$$

$$+ \frac{\kappa\theta \underline{c}^T \underline{b} - 1}{2/\eta - \kappa\theta^2 (\underline{c}^T \underline{b})^2 + 2\theta \underline{c}^T \underline{b}} \{P\underline{b}(\theta \underline{c}^T A + \underline{c}^T) + (\theta \underline{c}^T A + \underline{c}^T)^T \underline{b}^T P\}$$

$$+ \frac{1}{2/\eta - \kappa\theta^2 (\underline{c}^T \underline{b})^2 + 2\theta \underline{c}^T \underline{b}} (\Theta \underline{c}^T A + \underline{c}^T)^T (\Theta \underline{c}^T A + \underline{c}^T) \tag{5-128}$$

Therefore, the sufficient condition of (5-125) is

$$\hat{g} + 2/\eta = 2/\eta - \kappa\theta^2 (\underline{c}^T \underline{b})^2 + 2\theta \underline{c}^T \underline{b} > 0 \tag{5-129}$$

and

$$\hat{R} \leqq 0 \tag{5-130}$$

The condition (5-129) is assumed in (5-105). In order to obtain the frequency domain condition (5-104), let us apply Lemma 5-4 to (5-130). As $\hat{R}$ can be represented in the form of (5-117) where

$$\hat{\hat{M}} = \frac{\kappa}{1 + 2\kappa/\eta} (\theta \underline{c}^T A + \underline{c}^T)^T (\theta \underline{c}^T A + \underline{c}^T) + \mu \underline{c}\underline{c}^T$$

$$\hat{\hat{A}} = A$$

$$\hat{\hat{B}} = \underline{b}$$

$$\hat{\hat{C}} = \frac{1 - \kappa\theta \underline{c}^T \underline{b}}{1 + 2\kappa/\eta} (\theta \underline{c}^T A + \underline{c}^T)$$

$$\hat{\hat{K}} = \frac{2/\eta - \kappa\theta^2 (\underline{c}^T \underline{b})^2 + 2\theta \underline{c}^T \underline{b}}{1 + 2\kappa/\eta}$$

and all the other conditions of Lemma 5-4 are assumed in Theorem 5-5, there exists a positive-definite solution which satisfies $\hat{\hat{R}} = 0$ if and only if

$$\frac{2/\eta - \kappa\theta^2 (\underline{c}^T \underline{b})^2 + 2\theta \underline{c}^T \underline{b}}{1 + 2\kappa/\eta} + \frac{1 - \kappa\theta \underline{c}^T \underline{b}}{1 + 2\kappa/\eta} (\theta \underline{c}^T A + \underline{c}^T)(i\omega I - A)^{-1} \underline{b}$$

$$+ \underline{b}^T (-i\omega I - A)^{-1} (\theta \underline{c}^T A + \underline{c}^T)^T \frac{1 - \kappa\theta \underline{c}^T \underline{b}}{1 + 2\kappa/\eta}$$

$$- \frac{\kappa}{1 + 2\kappa/\eta} \underline{b}^T (-i\omega I - A^T)^{-1} (\theta \underline{c}^T A + \underline{c}^T)^T (\theta \underline{c}^T A + \underline{c}^T)(i\omega I - A)^{-1} \underline{b}$$

$$+ \mu \underline{b}^T (-i\omega I - A^T)^{-1} \underline{c}\underline{c}^T (i\omega I - A)^{-1} \underline{b} \geqq 0 \quad \text{for all } \omega \tag{5-132}$$

Using the identity

$$\underline{c}^T(\theta A + I)(sI - A)^{-1}\underline{b} = -\theta\underline{c}^T\underline{b} + (1 + \theta s)f(s)$$

in (5-132), we obtain (5-104) after some calculation. Thus, we have shown that if the conditions (5-104) and (5-105) are satisfied, there exists a positive-definite matrix P_j such that $\dot{v}_j|_{(5-119)}$ is estimated as (5-122).

Lastly, we examine the requirement (c) of Theorem 3-2. From (5-120) and (5-122), we obtain a test matrix given by (5-106). As all the requirements of Theorem 3-2 with $u_{0j} = 0$ are satisfied, System IIB is assured to be stable. This completes the first half of the proof.

Now, we will prove the last half of the theorem. In (5-130), if we set $\hat{R} = -2\varepsilon P$, (5-131) holds true with $\hat{\hat{A}} = A + \varepsilon I$. As all the eigenvalues of A have negative real parts, we can choose a small ε so that all the eigenvalues of $\hat{\hat{A}}$ have negative real parts. Therefore, from Lemma 5-4 the matrix equation $\hat{R} = -\varepsilon P$ has a positive-definite solution, if and only if the frequency domain condition (5-104) is satisfied with $s = i\omega - \varepsilon$. By using the solution of $\hat{R}_j = -2\varepsilon P_j$, $\dot{v}_j|_{(5-119)}$ can be estimated as

$$\dot{v}_j|_{(5-119)} \leq -\kappa_j u_{1j}^2 - \mu_j u_{2j}^2 - 2\varepsilon\underline{x}_j^T P_j \underline{x}_j$$

Therefore u_{0j} is given by $2\varepsilon\underline{x}_j^T P_j \underline{x}_j$ and from Theorem 3-2 System IIB is assured to be asymptotically stable in the large. This completes the proof. [Q.E.D.]

We can prove Theorem 5-6 in a parallel way to the proof of Theorem 5-5.

[Proof of Theorem 5-6] Let us define $v_j(\underline{x}_j, t)$ by

$$v_j(\underline{x}_j, t) = \underline{x}_j^T P_j \underline{x}_j + 2\int_0^{\underline{y}_j} \underline{\phi}_j^T(\underline{\sigma}) W_j^2 \Theta_j d\underline{\sigma} \tag{5-133}$$

where P_j is assumed to be a positive-definite matrix. Let us apply Theorem 3-2. For the system given by (3-1), the functions $\underline{\tilde{f}}_j$ and $\underline{\tilde{g}}_j$ are given by

$$\underline{\tilde{f}}_j(\underline{x}_j, t) = A_j\underline{x}_j - B_j\underline{\phi}_j(\underline{y}_j)$$

$$\underline{\tilde{g}}_j(\underline{x}, t) = B_j\underline{g}_j(\underline{y}_1, \cdots, \underline{y}_m, t)$$

and the isolated subsystem (3-6) turns out to be

$$\dot{\underline{x}}_j = A_j\underline{x}_j - B_j\underline{\phi}_j(\underline{y}_j), \quad \underline{y}_j = C_j\underline{x}_j \tag{5-134}$$

By (5-28) and (5-133), we obtain the requirement (b) of Theorem 3-2.

$$
\begin{aligned}
(\mathrm{grad}_j v_j)^T \tilde{\underline{g}}_j &= (2\underline{x}_j^T P_j + 2\underline{\phi}_j^T W_j^2 \Theta_j C_j) B_j \underline{g}_j \\
&\leq 2 |(\underline{x}_j^T P_j B_j + 2\underline{\phi}_j^T \Theta_j W_j^2 C_j B_j) W_j^{-1}| \sum_{k=1}^{m} W_j \beta_{jk} W_k^{-1} |W_k C_k \underline{x}_k| \\
&\leq 2 u_{1j}(\underline{x}_j) \sum_{k=1}^{m} \hat{\beta}_{jk} u_{2k}(\underline{x}_k) \qquad (5\text{-}135)
\end{aligned}
$$

where $\hat{\beta}_{jk}$ is given by (5-111) and the scalar functions $u_{1j}(\underline{x}_j)$ and $u_{2k}(\underline{x}_k)$ are defined by

$$
\begin{aligned}
u_{1j}(\underline{x}_j) &= ||\underline{x}_j^T P_j B_j W_j^{-1} + 2\underline{\phi}_j^T \Theta_j W_j^2 C_j B_j W_j^{-1}|| \\
u_{2j}(x_j) &= ||W_j C_j \underline{x}_j|| \qquad \text{for } j=1, \cdots, m \qquad (5\text{-}136)
\end{aligned}
$$

Next, let us examine the requirement (a) of Theorem 3-2. We will estimate $\dot{v}_j|_{(5\text{-}134)}$ in the form of (3-16) with $\lambda_j = 0$ and $u_{0j} = 0$; i.e.

$$
\dot{v}_j|_{(5\text{-}134)} \leq - \kappa_j ||\underline{x}_j^T P_j B_j W_j^{-1} + \underline{\phi}_j^T \Theta_j W_j^2 C_j B_j W_j^{-1}||^2 - \mu_j ||W_j C_j \underline{x}_j||^2 \qquad (5\text{-}137)
$$

While we investigate this requirement, we omit the suffix j which denotes the j-th subsystem (all the parameters need this suffix except for time t and the suffix k).

From (5-133) and (5-134)

$$
\dot{v}|_{(5\text{-}134)} = \underline{x}^T (PA + A^T P)\underline{x} - 2\underline{x}^T PB\underline{\phi} + 2\underline{\phi}^T W^2 \Theta C (A\underline{x} - B\underline{\phi}) \qquad (5\text{-}138)
$$

Since, by the assumption (5-26), $\phi_k(\underline{y})$ satisfies

$$
\phi_k(\underline{y})(\eta_k y_k - \phi_k(\underline{y})) \geq 0 \qquad \text{for } k=1, \cdots, s \qquad (5\text{-}139)
$$

(5-137) is satisfied if the right-hand side of (5-137) is greater than or equal to (5-138) for any nonlinear function $\underline{\phi}$ satisfying (5-139); i.e.

$$
\min_{\underline{\phi}} \ (\underline{\phi}^T \hat{G} \underline{\phi} + \underline{\phi}^T \hat{\underline{e}} + \hat{\underline{e}}^T \underline{\phi} + \hat{Q}) \geq 0 \qquad (5\text{-}140)
$$

subject to (5-139)

where

$$
\begin{aligned}
\hat{G} &= - \kappa \Theta W^2 CBW^{-2} B^T C^T W^2 \Theta + \Theta W^2 CB + B^T C^T W^2 \Theta \\
\hat{\underline{e}} &= (- \kappa \Theta W^2 CBW^{-2} B^T P + B^T P - W^2 \Theta CA)\underline{x} \qquad (5\text{-}141) \\
\hat{Q} &= - \underline{x}^T (\kappa PBW^{-2} B^T P + \mu WCC^T W + A^T P + PA)\underline{x}
\end{aligned}
$$

The next inequality is obviously satisfied.

$$\min_{\underline{\phi}} \quad (\underline{\phi}^T\hat{G}\underline{\phi} + \underline{\phi}^T\underline{\hat{e}} + \underline{\hat{e}}^T\underline{\phi} + \hat{Q})$$

subject to (5-139)

$$\geq \min_{\underline{\phi}} \{\underline{\phi}^T\hat{G}\underline{\phi} + \underline{\phi}^T\underline{\hat{e}} + \underline{\hat{e}}^T\underline{\phi} + \hat{Q} - 2\underline{\phi}^T W^2(C\underline{x} - \eta^{-1}\underline{\phi})\}$$

$$= \min_{\underline{\phi}} \{[\underline{\phi}^T(\hat{G} + 2\eta^{-1}W^2) + (\underline{\hat{e}} - W^2C\underline{x})(\hat{G} + 2\eta^{-1}W^2)^{-1}](\hat{G} + 2\eta^{-1}W^2)^{-1}$$
$$\cdot[(\hat{G} + 2\eta^{-1}W^2)\underline{\phi}^T + (\hat{G} + 2\eta^{-1}W^2)^{-1}(\underline{\hat{e}} - W^2C\underline{x})]$$
$$- (\underline{\hat{e}} - W^2C\underline{x})^T(\hat{G} + 2\eta^{-1}W^2)^{-1}(\underline{\hat{e}} - W^2C\underline{x}) + \hat{Q}\}$$

If we assume $\hat{G} + 2\eta^{-1}W^2 > 0$, then

$$\geq - (\underline{\hat{e}} - W^2C\underline{x})^T(\hat{G} + 2\eta^{-1}W^2)^{-1}(\underline{\hat{e}} - W^2C\underline{x}) + \hat{Q}$$

$$= - \underline{x}^T\hat{R}\underline{x} \qquad (5\text{-}142)$$

where $\hat{R}$ is a constant matrix given by

$$\hat{R} = (- \kappa\Theta\tilde{C}\tilde{B}\tilde{B}^TP + \tilde{B}^TP - \Theta\tilde{C}A - \tilde{C})^T(- \kappa\Theta\tilde{C}\tilde{B}\tilde{B}^T\tilde{C}^T\Theta + \Theta\tilde{C}\tilde{B} + \tilde{B}^T\tilde{C}^T\Theta + 2\eta^{-1})^{-1}$$
$$\cdot(- \kappa\Theta\tilde{C}\tilde{B}\tilde{B}^TP + \tilde{B}^TP - \Theta\tilde{C}A - \tilde{C}) + \kappa P\tilde{B}\tilde{B}^TP + \mu\tilde{C}\tilde{C}^T + A^TP + PA \qquad (5\text{-}143)$$

and $\tilde{B}$ and $\tilde{C}$ are given by (5-109).

Thus, the sufficient condition of (5-140) is

$$W^{-1}\{\hat{G} + 2\eta^{-1}W^2\}W^{-1} = 2\eta^{-1} - \kappa\Theta\tilde{C}\tilde{B}\tilde{B}^T\tilde{C}^T\Theta + \Theta\tilde{C}\tilde{B} + \tilde{B}^T\tilde{C}^T\Theta \geqq 0 \qquad (5\text{-}144)$$

and

$$\hat{R} \leqq 0 \qquad (5\text{-}145)$$

The condition (5-144) is assumed in (5-110). To obtain the frequency domain condition (5-108), let us apply Lemma 5-4 to (5-145). $\hat{R}$ can be represented in the form of (5-117) where

$$\hat{\hat{M}} = \hat{O} - \hat{S}^T\hat{E}^{-1}\hat{S}$$
$$\hat{\hat{A}} = A,\ \hat{\hat{B}} = \tilde{B},\ \hat{\hat{C}} = \hat{E}^{-1}\hat{S},\ \hat{\hat{K}} = \hat{E}^{-1} \qquad (5\text{-}146)$$

and

$$\hat{E} = \kappa I + (\kappa\tilde{B}^T\tilde{C}^T\Theta - I)(2\eta^{-1} - \kappa\Theta\tilde{C}\tilde{B}\tilde{B}^T\tilde{C}^T\Theta + \Theta\tilde{C}\tilde{B} + \tilde{B}^T\tilde{C}^T\Theta)^{-1}(\Theta\tilde{C}\tilde{B}\kappa - I)$$

$$\hat{S} = - (\kappa\tilde{B}^T\tilde{C}^T\Theta - I)(2\eta^{-1} - \kappa\Theta\tilde{C}\tilde{B}\tilde{B}^T\tilde{C}^T\Theta + \Theta\tilde{C}\tilde{B} + \tilde{B}^T\tilde{C}^T\Theta)^{-1}(\Theta\tilde{C}A + \tilde{C})$$

$$\hat{O} = \mu\tilde{C}^T\tilde{C} + (A^T\tilde{C}^T\Theta + \tilde{C}^T)(2\eta^{-1} - \kappa\Theta\tilde{C}\tilde{B}\tilde{B}^T\tilde{C}^T\Theta + \Theta\tilde{C}\tilde{B} + \tilde{B}^T\tilde{C}^T\Theta)^{-1}(\Theta\tilde{C}A + \tilde{C})$$
$$(5\text{-}147)$$

As κ is a positive constant and (5-144) is assumed, $\hat{E}$ is positive-definite and, so, $\hat{\hat{K}}$ (= $\hat{E}^{-1}$) is invertible. Since $\hat{\hat{M}}$ can be represented as

$$\hat{\hat{M}} = \kappa\hat{X}^T\hat{H}^{-1/2}\{I - (I + \frac{1}{\kappa}\hat{D}^T\hat{H}^{-1}\hat{D})^{-1}\}\hat{H}^{-1/2}\hat{X}$$

where $\hat{D}$ and $\hat{H}$ are given by (5-109) and $\hat{X}$ is defined by $\hat{X} = \tilde{C} + \Theta\tilde{C}A$, $\hat{\hat{M}}$ is positive-semi-definite (Note that $\hat{H}$ is positive-definite from (5-110)). As $\kappa\tilde{B}^T\tilde{C}^T\Theta - I$ is non-singular from (5-110) and $\hat{\hat{C}}$ is given by (5-146), $(\hat{A}, \hat{C})$ is observable if $(C + \Theta CA, A)$ is observable. Other requirements of Lemma 5-4 are clearly satisfied. By substituting (5-146) into (5-116) and using the identity

$$(\tilde{C} + \Theta\tilde{C}A)\tilde{C}(sI - A)^{-1}\tilde{B} = -\Theta\tilde{C}\tilde{B} + (I + s\Theta)\tilde{F}(s)$$

we obtain (5-108). Thus we have shown that if the conditions (5-108) and (5-110) are satisfied, there exists a positive-definite matrix P_j such that $\dot{v}_j|_{(5\text{-}134)}$ is estimated as (5-137).

Lastly, we examine the requirement (c) of Theorem 3-2. From (5-135) and (5-137), we can immediately obtain the test matrix given by (5-111). As all the requirements of Theorem 3-2 with $u_{0j} = 0$ are satisfied, System IID is stable. This completes the first half of the proof. As for the proof of asymptotic stability, we can prove in a parallel way to the proof of Theorem 5-5.

[Q.E.D.]

Sec. 5.4. Remarks on Theorem 5-5

In this section, we clarify the relation between Theorem 5-3 and Theorem 5-5 and the relation between Theorem 5-5 and the L_2-stability condition (Saeki et al. 1979).

To compare Theorem 5-5 with Theorem 5-3, we assume that $\zeta_j = 0$, $\theta_j = 0$ and all the eigenvalues of A_j have negative real parts. By applying Lemma 4-3, the positive-realness of $h_j(s)$ of Theorem 5-3 can be expressed as

$$1 + \frac{\eta_j}{2} f_j^*(s) + \frac{\eta_j}{2} f_j(s) + \{(\frac{\eta_j}{2})^2 - \kappa_j\mu_j\}f_j^*(s)f_j(s) \geq 0$$

$$\text{for } s = i\omega \quad 0 \leq \omega \leq \infty \qquad (5\text{-}147)$$

On the other hand, from (5-104) of Theorem 5-5, we obtain

$$1 + \frac{\eta_j}{2} f_j^*(s) + \frac{\eta_j}{2} f_j(s) + \{(\frac{\eta_j}{2})^2 - (\frac{\eta_j}{2} + \kappa_j)(\frac{\eta_j}{2} + \mu_j)\}f_j^*(s)f_j(s) \geq 0$$

$$\text{for } s = i\omega \quad 0 \leq \omega \leq \infty \qquad (5\text{-}148)$$

From the above two equations, (5-148) is satisfied with $\kappa_j = \kappa_{0j} - \eta_j/2$, $\mu_j = \mu_{0j} - \eta_j/2$, if (5-147) is satisfied with $\kappa_j = \kappa_{0j}$ and $\mu_j = \mu_{0j}$. In this case, the diagonal elements of the test matrices of Theorem 5-3 and Theorem 5-5 are given by

$$\sqrt{\kappa_{0j}\mu_{0j}} - \eta_j/2 - \beta_{jj} \qquad \text{(from (5-78))}$$

$$\sqrt{(\kappa_{0j} - \eta_j/2)(\mu_{0j} - \eta_j/2)} - \beta_{jj} \qquad \text{(from (5-106))}$$

respectively, and the off-diagonal elements of the matrices are the same. In Theorem 5-3, the values of κ_j and μ_j can be chosen arbitrarily without affecting the M-matrix condition as long as the product $\kappa_j\mu_j$ is fixed. On the other hand, the diagonal elements of Theorem 5-5 attain the largest value for $\kappa_{oj} = \mu_{oj}$, and in this case the test matrix of Theorem 5-5 agrees with that of Theorem 5-3. Thus, we have shown that Theorem 5-5 with $\theta_j = 0$ becomes as sharp as Theorem 5-3 by choosing $\kappa_j = \mu_j$ in Theorem 5-5. So, Theorem 5-5 is potential to give a sharper result than Theorem 5-3 because of arbitrary parameters θ_j.

Next let us compare Theorem 5-5 with the L_2-stability condition of Saeki et al. (1979). By applying Theorem 2 of Saeki et al. (1979) to System IIB, we obtain the next theorem.

Theorem 5-6

System IIB is L_2-stable if the leading principal minors of $\hat{A} - \hat{B}$ are all positive for some $\theta_1 > 0, \cdots, \theta_m > 0$ where $\hat{A} = \mathrm{diag}(\alpha_j)$, α_j's are positive constants such that the Nyquist diagram of each $f_j(i\omega)(1 + i\omega\theta_j)$ lies inside the disc with center

$$\frac{1}{2}\left(\frac{1}{\alpha_j} - \frac{1}{\eta_j + \alpha_j}\right) + 0i$$

and radius

$$\frac{1}{2}\left|\frac{1}{\alpha_j} + \frac{1}{\eta_j + \alpha_j}\right|$$

and $\hat{B}$ is given by $\hat{B} = (\beta_{jk})$.

From Remark 5-5 and the equivalent relation between Theorem 5-2 and Theorem 5-4, Theorem 5-6 can be represented as follows.

Theorem 5-7 (Alternative form of Theorem 5-6)

Assume that A_j is stable. System IIB is L_2-stable if there exist positive numbers κ_j and μ_j and a constant θ_j for each j such that

$$1 + \frac{\eta_j}{2}\hat{f}_j^*(s) + \frac{\eta_j}{2}\hat{f}_j(s) + \{(\frac{\eta_j}{2})^2 - \sqrt{\kappa_j\mu_j}\}\hat{f}_j^*(s)\hat{f}_j(s) \geqq 0$$

$$\text{for } s = i\omega \quad 0 \leqq \omega < \infty \tag{5-149}$$

where

$$\hat{f}_j(s) = (1 + \theta_j s)f_j(s) \tag{5-150}$$

and if the mxm matrix $\hat{A} - \hat{B}$ is an M-matrix where

$$\hat{A} = \mathrm{diag}(\sqrt{\kappa_j\mu_j} - \eta_j/2), \quad \hat{B} = (\beta_{jk}) \tag{5-151}$$

The frequency domain condition (5-104) of Theorem 5-5 is represented as

$$1 + \frac{\eta_j}{2}\hat{f}_j^*(s) + \frac{\eta_j}{2}\hat{f}_j(s) - \frac{\kappa_j\eta_j}{2}\hat{f}_j^*(s)\hat{f}_j(s) - \mu_j\{\kappa_j + \frac{\eta_j}{2}\}\frac{\hat{f}_j^*(s)\hat{f}_j(s)}{|1 + \theta_j s|^2} \geqq 0$$

$$\text{for } s = i\omega \quad 0 \leqq \omega < \infty \tag{5-152}$$

As $|1 + \theta_j s|^2 \geqq 1$ for $s = i\omega$, the sufficient condition of (5-152) is that

$$1 + \frac{\eta_j}{2}\hat{f}_j^*(s) + \frac{\eta_j}{2}\hat{f}_j(s) + \{(\frac{\eta_j}{2})^2 - (\frac{\eta_j}{2} + \kappa_j)(\frac{\eta_j}{2} + \mu_j)\}\hat{f}_j^*(s)\hat{f}_j(s) \geqq 0$$

$$\text{for } s = i\omega \quad 0 \leqq \omega < \infty \tag{5-153}$$

As (5-149) and (5-153) can be obtained by setting $f_j = \hat{f}_j$ in (5-147) and (5-148), respectively, we can conclude that Theorem 5-5 with $\kappa_j = \mu_j$ is sharper than Theorem 5-7 (equivalently, Theorem 5-6).

In this chapter, we consider the situation such that the description of the systems given in Fig. 1.1 and Fig. 1.2 are valid only for some finite region. We estimate the stability region by the Lyapunov functions which were used in the proofs of Chapter 4 and Capter 5.

Sec. 6.1. Stability Region of Systems with Multiple Nonlinear Feedbacks

In this section, we assume that the nonlinear function $\phi_j(\underline{y}, t)$ of System I satisfies (4-5) only for the values of y_j restricted by

$$-\rho_{1j} \leqq y_j \leqq \rho_{2j} \qquad \text{for } j=1, \cdots, m \tag{6-1}$$

where ρ_{1j} and ρ_{2j} are positive constants. Let us study the estimate of the stability region surrounding the origin.

Theorem 6-1 shows a concrete way of constructing a Lyapunov function of System I and Theorem 6-2 gives a method of estimating the stability region of System I.

Theorem 6-1 (Construction of Lyapunov function of System I)

Assume that the requirements of Theorem 4-2 are satisfied. Then, there exists a positive-definite nxn matrix P which satisfies

$$PB\hat{R}B^TP + P\hat{A} + \hat{A}^TP + \hat{Q} = 0 \tag{6-2}$$

where

$$\begin{aligned} \hat{R} &= \hat{W}^{-1}\tilde{\eta}\{2I + \hat{W}D\hat{W}^{-1}\tilde{\eta} + \tilde{\eta}\hat{W}^{-1}D^T\hat{W}\}^{-1}\tilde{\eta}\hat{W}^{-1} \\ \hat{A} &= A - B\tilde{\eta}\{2I + D\hat{W}^{-2}\tilde{\eta} + \tilde{\eta}\hat{W}^{-2}D^T\}^{-1}\tilde{C} \\ \hat{Q} &= \tilde{C}^T\hat{W}\{2I + D\hat{W}^{-2}\tilde{\eta} + \tilde{\eta}\hat{W}^{-2}D^T\}^{-1}\hat{W}\tilde{C} \end{aligned} \tag{6-3}$$

$$\tilde{C} = C - D\zeta C, \quad \tilde{A} = A - B\zeta C, \quad \tilde{\eta} = \eta - \zeta \tag{6-4}$$

and, in each case, $\hat{W}$ is given by

$$\begin{aligned} &\text{(i)} \quad \text{in Case 1, } \hat{W} = \zeta^{1/2}\eta^{1/2}RW \\ &\text{(ii)} \quad \text{in Case 2, } \hat{W} = W \\ &\text{(iii)} \quad \text{in Case 3, } \hat{W} = (-\zeta)^{1/2}\eta^{1/2}RW \end{aligned} \tag{6-5}$$

and the function $v(\underline{x})$ defined by

$$v(\underline{x}) = \underline{x}^TP\underline{x} \tag{6-6}$$

is a Lyapunov function of System I.

Theorem 6-2 (Stability region of System I)

Assume that the requirements of Theorem 4-2 and the restriction (6-1) are satisfied. Let P and $v(\underline{x})$ be as given in Theorem 6-1 and let $\Theta = (\theta_{jk})$ be given by

$$\theta_{jk} = |(D)_{jk}|\max(|\zeta_k|, \eta_k) \quad \text{for } j, k = 1, \cdots, m \qquad (6\text{-}7)$$

If $I - \Theta$ is an M-matrix, then the region

$$S = \{\underline{x} \mid v(\underline{x}) \leqq \pi_0\} \qquad (6\text{-}8)$$

is included in the stability region of the origin of System I where

$$\pi_0 = \min_{j=1, \cdots, m} \pi_j \qquad (6\text{-}9)$$

$$\pi_j = \min_{\underline{c}_j^T \underline{x} = \tilde{\rho}_j} \underline{x}^T P \underline{x} = \frac{\tilde{\rho}_j^2}{\underline{c}_j^T P^{-1} \underline{c}_j} \quad j=1, \cdots, m, \qquad (6\text{-}10)$$

$\underline{c}_j$ is the j-th row vector of the matrix C, and the positive constants $\tilde{\rho}_j$ satisfy

$$[I - \Theta]^{-1}\begin{pmatrix}\tilde{\rho}_1 \\ \vdots \\ \tilde{\rho}_m\end{pmatrix} \leqq \begin{pmatrix}\rho_1 \\ \vdots \\ \rho_m\end{pmatrix} \qquad (6\text{-}11)$$

$$\rho_j = \min(\rho_{1j}, \rho_{2j}) \qquad (6\text{-}12)$$

Remark 6-1

In Theorem 6-1, the algebraic equation (6-2) has the form of

$$P\overline{A}P + P\overline{B} + \overline{B}^T P + \overline{C} = 0 \qquad (6\text{-}13)$$

We can apply the Potter's method (1966) to solve this equation. But it must be noted that there are, in general, many positive-definite solutions. Any one of such solutions can be used to construct a Lyapunov function. The problem of choosing P which gives the largest estimate of the stability region is open.

Remark 6-2

In order to obtain a larger estimate by using Theorem 6-2, we need to choose the value of $\tilde{\rho}_j$ properly. This problem is to maximize the value of π_0

subject to (6-11). If we define $\hat{z}_j$ as

$$\hat{z}_j = \frac{\rho_j}{(\underline{c}_j^T P^{-1} \underline{c}_j)^{1/2}} ,$$

then the problem is formulated as the following maximization problem:

$$\text{Obtain} \quad \pi_0^{1/2} = \max_{\hat{\underline{z}}} \min_j \hat{z}_j$$

subject to the constraints

$$\begin{pmatrix} & & \\ & E & \\ & & \end{pmatrix} \begin{pmatrix} \hat{z}_1 \\ \vdots \\ \hat{z}_m \end{pmatrix} \leqq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and

$$\hat{z}_j > 0 \qquad \text{for } j=1, \cdots, m$$

where

$$E = \begin{pmatrix} \rho_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \rho_m^{-1} \end{pmatrix} [I - \Theta]^{-1} \begin{pmatrix} (\underline{c}_1^T P^{-1} \underline{c}_1)^{1/2} & & 0 \\ & \ddots & \\ 0 & & (\underline{c}_m^T P^{-1} \underline{c}_m)^{1/2} \end{pmatrix}$$

$$E \triangleq (e_{ij}), \; e_{ij} \geqq 0 \text{ for all } i, j$$

This problem is a min max problem with constraints. The method of Demyanov & Malozemov (1974) would be helpful to solve this problem. In this thesis, we do not investigate this problem further.

Theorem 6-1 is directly obtained from the proof of Theorem 4-2. So, the proof of Theorem 6-1 is omitted.

[Proof of Theorem 6-2] From the assumption of Theorem 4-2, we set $\underline{u}=\underline{0}$. Then, from (4-2) and (4-3), we obtain the next algebraic equation with respect to $\underline{y}$.

$$\underline{y} = C\underline{x} - D\underline{\phi}(\underline{y}, t) \tag{6-14}$$

From (6-1) and (6-12), y_j satisfies

$$|y_j| \leq \rho_j \qquad \text{for } j=1, \cdots, m \tag{6-15}$$

From the proof of Theorem 4-2, the time-derivative of v along the solution of System I is negative-semi-definite for such y_j that satisfies (6-15). Therefore, the set S given by

$$S = \{\underline{x} \mid v(\underline{x}) \leq \min_{\underline{x}\in\partial R_0} v(\underline{x})\} \tag{6-16}$$

is contained in the stability region where the set R_0 is the set of $\underline{x}$ which satisfies both (6-14) and (6-15). Now, let us estimate R_0. From (4-5)

$$|\phi_j(\underline{y}, t)| \leq \max(\eta_j, |\zeta_j|)|y_j| \tag{6-17}$$

From (6-7), (6-14) and (6-17)

$$\begin{pmatrix} |\underline{c}_1^T\underline{x}| \\ \vdots \\ |\underline{c}_m^T\underline{x}| \end{pmatrix} \geq [I - \Theta] \begin{pmatrix} |y_1| \\ \vdots \\ |y_m| \end{pmatrix} \tag{6-18}$$

As $I - \Theta$ is an M-matrix, we obtain

$$[I - \Theta]^{-1} \begin{pmatrix} |\underline{c}_1^T\underline{x}| \\ \vdots \\ |\underline{c}_m^T\underline{x}| \end{pmatrix} \geq \begin{pmatrix} |y_1| \\ \vdots \\ |y_m| \end{pmatrix} \tag{6-19}$$

From (6-11) and (6-19), if $|\underline{c}_j^T\underline{x}| \leq \tilde{\rho}_j$ is satisfied for $j=1, \cdots, m$, then $|y_j| \leq \rho_j$ is satisfied for $j=1, \cdots, m$. Therefore, an estimate of R_0 is given by

$$\tilde{R}_0 = \{\underline{x} \mid |\underline{c}_j^T\underline{x}| \leq \tilde{\rho}_j, \quad j=1, \cdots, m\} \tag{6-20}$$

As $\tilde{R}_0$ is included in R_0, we may use $\tilde{R}_0$ instead of R_0 in (6-16). It is obviously satisfied that $\pi_0 = \min_{\underline{x}\in\partial\tilde{R}_0} v(\underline{x})$. This completes the proof. [Q.E.D.]

Sec. 6.2. Stability Region of Composite Systems Consisting of Subsystems with Nonlinear Feedbacks

Let us consider the stability region estimate of System IIA - IID. In this section, we assume that the nonlinear functions ϕ_j of System IIA and System IIB satisfy (5-4) and (5-10), respectively, only for the values of y_j restricted by

$$-\rho_{1j} \leq y_j \leq \rho_{2j} \qquad \text{for } j=1, \cdots, m \tag{6-21}$$

where ρ_{1j} and ρ_{2j} are positive constants. And similarly we assume that the nonlinear functions $\underline{\phi}_j$ of System IIC and System IID satisfy (5-17) and (5-26), respectively, only for the values of $\underline{y}_j$ restricted by

$$-\rho_k^{(1j)} \leq y_k^{(j)} \leq \rho_k^{(2j)} \qquad \text{for } j=1, \cdots, m \quad k=1, \cdots, s_j \tag{6-22}$$

where $\rho_k^{(1j)}$ and $\rho_k^{(2j)}$ are positive constants. In these situations we will estimate the stability regions.

6.2.1. Construction of Lyapunov Functions

We obtain the following theorems to construct Lyapunov functions of System IIA - IID from the proofs of Theorems 5-2, 5-5, 5-3, and 5-6, respectively.

Theorem 6-3 (Construction of Lyapunov function of System IIA)

Assume that the requirements of Theorem 5-2 are satisfied, and let δ be a sufficiently small positive number which satisfies (5-49). Then, there exists a positive-definite $n_j x n_j$ natrix P_j which satisfies

$$(\eta_j - \zeta_j + 2\alpha_j - 2\delta)^2 P_j \underline{b}_j \underline{b}_j^T P_j + P_j(2A_j - (\eta_j + \zeta_j)\underline{b}_j \underline{c}_j^T)$$
$$+ (2A_j - (\eta_j + \zeta_j)\underline{b}_j \underline{c}_j^T)^T P_j + \underline{c}_j \underline{c}_j^T + \varepsilon_j I = 0 \tag{6-23}$$

for sufficiently small number ε_j, there exists a diagonal matrix $D = \mathrm{diag}(d_j)$ with $d_j > 0$ which makes $\hat{M}D\hat{M} - \hat{\Gamma}^T D\hat{\Gamma}$ positive-semi-definite where

$$\hat{M} = \mathrm{diag}\left(\frac{\eta_j - \zeta_j}{2} + \alpha_j - \delta \right), \quad \hat{\Gamma} = \mathrm{diag}\left(\frac{\eta_j - \zeta_j}{2} \right) + \hat{B} \tag{6-24}$$

and the function $v(\underline{x})$ defined by

$$v(\underline{x}) = \sum_{j=1}^{m} d_j \underline{x}_j^T P_j \underline{x}_j \tag{6-25}$$

is a Lyapunov function of System IIA.

Theorem 6-4 (Construction of Lyapunov function of System IIC)

Assume the requirements of Theorem 5-3 are satisfied and let ε be a sufficiently small positive number which makes $H_j(s - \varepsilon)$ positive-real. Then, there exists a positive-definite $n_j x n_j$ matrix P_j which satisfies

$$P_j B_j W_j^{-1}(\tilde{\eta}_j - \tilde{\zeta}_j)W_j^{-1} B_j^T P_j + P_j[2A_j - \varepsilon_j I - B_j(\tilde{\eta}_j + \tilde{\zeta}_j)C_j]$$
$$+ [2A_j - \varepsilon_j I - B_j(\tilde{\eta}_j + \tilde{\zeta}_j)C_j]^T P_j + C_j^T W_j^2 C_j = 0 \tag{6-26}$$

$$\text{for } j=1, \cdots, m$$

where

$$\tilde{\eta}_j = \eta_j + \alpha_j I , \quad \tilde{\zeta}_j = \zeta_j - \alpha_j I \tag{6-27}$$

there exists a diagonal matrix $D = \mathrm{diag}(d_j)$ with $d_j > 0$ which makes $\hat{M}D\hat{M} - \hat{\Gamma}^T D\hat{\Gamma}$ positive-semi-definite where

$$\hat{M} = \mathrm{diag}[\max_k(\eta_k^{(j)} - \zeta_k^{(j)} + 2\alpha_j)]$$

$$\hat{\Gamma} = \mathrm{diag}[\max_k(\eta_k^{(j)} - \zeta_k^{(j)})] + \hat{B} \tag{6-28}$$

and the function $v(\underline{x})$ defined by

$$v(\underline{x}) = \sum_{j=1}^{m} d_j \underline{x}_j^T P_j \underline{x}_j \tag{6-29}$$

is a Lyapunov function of System IIC.

Theorem 6-5 (Construction of Lyapunov function of System IIB)

Assume that the requirements of Theorem 5-5 are satisfied. Then, there exists a positive-definite $n_j \times n_j$ matrix P_j which satisfies

$$\begin{aligned}
&\left\{ \frac{(\kappa_j\theta_j\underline{c}_j^T\underline{b}_j - 1)^2}{2/\eta_j - \kappa_j\theta_j^2(\underline{c}_j^T\underline{b}_j)^2 + 2\theta_j\underline{c}_j^T\underline{b}_j} + \kappa_j \right\} P_j\underline{b}_j\underline{b}_j^T P_j \\
&\quad + P_j\left\{A_j + \frac{\kappa_j\theta_j\underline{c}_j^T\underline{b}_j - 1}{2/\eta_j - \kappa_j\theta_j^2(\underline{c}_j^T\underline{b}_j)^2 + 2\theta_j\underline{c}_j^T\underline{b}_j}\underline{b}_j(\theta_j\underline{c}_j^T A_j + \underline{c}_j^T)\right\} \\
&\quad + \left\{A_j + \frac{\kappa_j\theta_j\underline{c}_j^T\underline{b}_j - 1}{2/\eta_j - \kappa_j\theta_j^2(\underline{c}_j^T\underline{b}_j)^2 + 2\theta_j\underline{c}_j^T\underline{b}_j}\underline{b}_j(\theta_j\underline{c}_j^T A_j + \underline{c}_j^T)\right\}^T P_j \\
&\quad + \mu_j\underline{c}_j^T\underline{c}_j = 0 \qquad \text{for } j=1, \cdots, m
\end{aligned} \tag{6-30}$$

there exists a diagonal matrix $D = \mathrm{diag}(d_j)$ with $d_j > 0$ which makes $\hat{M}D\hat{M} - \hat{\Gamma}^T D\hat{\Gamma}$ positive-semi-definite where

$$\hat{M} = \mathrm{diag}(\mu_j^{1/2}), \quad \hat{\Gamma} = \mathrm{diag}(\kappa_j^{-1/2})\hat{B} \tag{6-31}$$

and the function $v(\underline{x})$ defined by

$$v(\underline{x}) = \sum_{j=1}^{m} d_j\left\{ \underline{x}_j^T P_j \underline{x}_j + 2\theta_j \int_0^{\underline{c}_j^T\underline{x}_j} \phi_j(\sigma)\,d\sigma \right\} \tag{6-32}$$

is a Lyapunov function of System IIB.

Theorem 6-6 (Construction of Lyapunov function of System IID)

Assume that the requirements of Theorem 5-6 are satisfied. Then, there exists a positive-definite $n_j x n_j$ matrix P_j which satisfies

$$P_j\tilde{B}_j\{\kappa_j I + \hat{D}_j^T\hat{H}_j^{-1}\hat{D}_j\}\tilde{B}_j^T P_j + P_j\{A_j + \tilde{B}_j\hat{D}_j^T\hat{H}_j^{-1}(\Theta_j\tilde{C}_j A_j + \tilde{C}_j)\}$$

$$+ \{A_j + \tilde{B}_j\hat{D}_j^T\hat{H}_j^{-1}(\Theta_j\tilde{C}_j A_j + \tilde{C}_j)\}^T P_j + \mu_j\tilde{C}_j\tilde{C}_j^T$$

$$+ (\Theta_j\tilde{C}_j A_j + \tilde{C}_j)^T\hat{H}_j^{-1}(\Theta_j\tilde{C}_j A_j + \tilde{C}_j) = 0 \quad \text{for } j=1, \cdots, m \tag{6-33}$$

there exists a diagonal matrix $D = \text{diag}(d_j)$ with $d_j>0$ which makes $\hat{M}D\hat{M} - \hat{\Gamma}^T D\hat{\Gamma}$ positive-semi-definite where

$$\hat{M} = \text{diag}(\mu_j^{1/2}),\ \hat{\Gamma} = \text{diag}(\kappa_j^{-1/2})\hat{B} \tag{6-34}$$

and the function $v(\underline{x})$ defined by

$$v(\underline{x}) = \sum_{j=1}^{m} d_j\{\ \underline{x}_j^T P_j \underline{x}_j + 2\int_{\Omega}^{C_j\underline{x}_j} \underline{\Phi}_j^T(\underline{\sigma}) W_j^2 \Theta_j d\underline{\sigma}\} \tag{6-35}$$

is a Lyapunov function of System IID.

Remark 6-3

Remark 6-1 is also applied to eqs. (6-23), (6-26), (6-30), and (6-33). We can determine the values of d_j by the method stated in 3.4.2.

6.2.2. Estimate of Stability Region

In 6.2.1, we presented the method of constructing Lyapunov functions of System IIA - IID. The following theorems give the estimates of the stability regions of System IIA - IID by using these Lyapunov functions.

Theorem 6-7 (Stability region estimate of System IIA)

Assume the requirements of Theorem 5-2 and (6-21) are satisfied and let P_j, D and v(x) be as given in Theorem 6-3. Then the region

$$S_x = \{\underline{x} \mid v(\underline{x}) \leqq \tilde{\pi}_0\} \tag{6-36}$$

is included in the stability region of the origin of System IIA where

$$\tilde{\pi}_0 = \min_{1\leqq j\leqq m} d_j v_{0j} \tag{6-37}$$

$$v_{0j} = \min\left(\frac{\rho_{1j}^2}{\underline{c}_j^T P_j^{-1} \underline{c}_j} , \frac{\rho_{2j}^2}{\underline{c}_j^T P_j^{-1} \underline{c}_j} \right) \tag{6-38}$$

Theorem 6-8 (Stability region estimate of System IIC)

Assume the requirements of Theorem 5-3 and (6-22) are satisfied and let P_j, D and v(x) be as given in Theorem 6-4. Then, the region S_x given by (6-36) is included in the stability region of the origin of System IIC where $\tilde{\pi}_0$ is given by (6-37) and v_{0j} is given by

$$v_{0j} = \min_{1 \leq k \leq \ell_j} \{\min\left(\frac{(\rho_k^{(1j)})^2}{(\underline{c}_k^{(j)})^T P_j^{-1} \underline{c}_k^{(j)}} , \frac{(\rho_k^{(2j)})^2}{(\underline{c}_k^{(j)})^T P_j^{-1} \underline{c}_k^{(j)}} \right)\} \tag{6-39}$$

where $\underline{c}_k^{(j)}$ is the k-th row vector of C_j.

Theorem 6-9 (Stability region estimate of System IIB)

Assume the requirements of Theorem 5-4 and (6-21) are satisfied and let P_j, D and v(x) be as given in Theorem 6-5. The resion S_x given by (6-36) is included in the stability region of the origin of System IIB where $\tilde{\pi}_0$ is given by (6-37) and v_{0j} is given by

$$v_{0j} = \min\{ \frac{(\rho_{1j})^2}{\underline{c}_j^T P_j^{-1} \underline{c}_j} + 2\theta_j \int_0^{-\rho_{1j}} \phi_j(\sigma) d\sigma, \frac{(\rho_{2j})^2}{\underline{c}_j^T P_j^{-1} \underline{c}_j} + 2\theta_j \int_0^{-\rho_{2j}} \phi_j(\sigma) d\sigma\} \tag{6-40}$$

Theorem 6-10 (Stability region estimate of System IID)

Assume the requirements of Theorem 5-5 and (6-22) are satisfied and let P_j, D and v(x) be as given in Theorem 6-6. The region S_x given by (6-36) is included in the stability region of the origin of System IID where $\tilde{\pi}_0$ is given by (6-37) and v_{0j} is given by

$$v_{0j} = \min_{1 \leq k \leq \ell_j} \{ \min_{\underline{x}_j} v_j(\underline{x}_j) \text{ subject to } (\underline{c}_k^{(j)})^T \underline{x}_j = -\rho_k^{(1j)} \text{ or } \rho_k^{(2j)} \} \tag{6-41}$$

The concrete method of computing v_{0j} is given by the next lemma.

Lemma 6-1 (Pai and Narayana 1976)

Consider a typical hyperplane defined by

$$\underline{c}_k^{(j)} \underline{x}_j = \rho \tag{6-42}$$

where $\underline{c}_k^{(j)}$ is the k-th row vector of C_j. Then, the minimum value of

$$v_j(\underline{x}_j) = \underline{x}_j^T P_j \underline{x}_j + 2 \int_0^{C_j \underline{x}_j} \underline{\Phi}_j^T(\underline{\sigma}) W_j^2 \Theta_j d\underline{\sigma} \tag{6-43}$$

under the constraint (6-42) is given by

$$v_j(\hat{\underline{x}}_j) = \hat{\underline{y}}_j^T [(C_j P_j^{-1} C_j^T)^{-1}] \hat{\underline{y}}_j + 2 \int_0^{\hat{\underline{y}}_j} \underline{\Phi}_j W_j^2 \Theta_j d\underline{\sigma} \tag{6-44}$$

In the above equation, $\hat{\underline{y}}_j = C_j \hat{\underline{x}}_j$ and $\hat{\underline{x}}_j$ is given by solving the next nonlinear algebraic equations in the unknowns $\underline{x}_j$, λ_k and σ_j (j=1, ···, s_j, j≠k)

$$\begin{pmatrix} 2P_j & -C_j^T \\ C_j & 0 \end{pmatrix} \begin{pmatrix} \underline{x}_j \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} -2W_j^2 \Theta_j \underline{\Phi}_j(\underline{\sigma}) \\ \underline{\sigma} \end{pmatrix} \tag{6-45}$$

where

$$\begin{aligned} \underline{\lambda} &= [0, \cdot \cdot \cdot, 0, \lambda_k, 0, \cdot \cdot \cdot, 0]^T \\ \underline{\sigma} &= [\sigma_1, \cdots, \sigma_{k-1}, \rho, \sigma_{k+1}, \cdots, \sigma_{s_j}]^T \end{aligned} \tag{6-46}$$

[Proof of Theorem 6-7] Let us apply Theorem 3-3. From the assumption (6-21), the set R_j in Theorem 3-3 is given by

$$R_j = \{\underline{x}_j \mid \underline{x}_j \text{ satisfies } -\rho_{1j} \leq \underline{c}_j^T \underline{x}_j \leq \rho_{2j}\} \tag{6-47}$$

and the description is valid in the set $R = R_1 \times \cdots \times R_m$. As v_j is a positive-definite function and v_{0j} is given by (6-38), S_j given by (3-29) is clearly contained in R_j. This completes the proof. [Q.E.D.]

The other three theorems can be proved in parallel to the above proof. To derive (6-40), we applied the result of Walker & McClamroch (1967). We can obtain larger estimates than the estimates obtained by the above theorems, respectively, by means of the open Lyapunov surface (J.L. Willems 1969).

Chapter 7 Examples

In this chapter, we examine eight examples. The contents are outlined as follows.

Example 1

The well-posedness condition (Theorem 4-1) is applied to a 2-input 2-output system of type I. In order to calculate the uniform instantaneous gains, Theorem 2-1 is also used.

Example 2

Theorem 3-1 and Theorem 3-2 are applied to the transient stability analysis of a multi-machine power system. By using the theorems, we obtain the stability conditions and construct Lur'e type Lyapunov functions, respectively. Sharpness of the two stability conditions are compared with each other.

Example 3

By applying Theorem 3-3 and Weissenberger's method, we obtain two estimates of the stability region of a composite system which is composed of 3 subsystems. The largeness of the two estimates are compared with each other.

Example 4

The method of searching a feasible weight, which was proposed in Sec. 4.4, is applied to a 3-input 3-output system of type I. This system cannot be assured to be stable by both M-matrix condition (Theorem 4-3) and Rosenbrock's criteria (Theorem 4-5).

Example 5

The weighted multivariable circle criterion (Theorem 4-2) together with the method of searching a feasible weight (Sec. 4.4), the M-matrix condition (Theorem 4-3), Rosenbrock's criteria (Theorem 4-5), and Hurwitz condition are applied to a 2-input 2-output system. System parameter regions in which stability was assured by the four conditions are illustrated.

Example 6

The circle criterion type condition (Theorem 5-2) is applied to a system which is composed of three subsystems. We construct a quadratic form Lyapunov function by using Theorem 6-3 and obtain the estimate of the stability region by using Theorem 6-7. We also apply the Weissenberger's method (1973) and obtain parallel results to the above. These two methods are compared with each other.

Example 7

In Remark 5-5, we have examined how to find the feasible value of Θ_j which is an arbitrary parameter contained in Theorem 5-6. We calculate the " optimum " value of Θ_j for a simple system to investigate the influence of Θ_j on the condition of Theorem 5-6.

Example 8

Popov criterion type condition (Theorem 5-5) is applied to the multi-machine power system which was examined in Example 2. We construct a Lur'e type Lyapunov function by using Theorem 6-5 and obtain the estimate of the stability region by using Theorem 6-9. We also apply the method of Araki et al. (1980) to this system and obtain the parallel results to the above.

Sec. 7.1. Example 1 (Well-posedness)

Consider a dynamical system in a block-diagram form as shown in Fig. 7-1 where g_{11}, g_{12}, g_{21}, and g_{22} are the plant models, and ϕ_1 and ϕ_2 are the gain controller models which include the saturation of the actuators. Consider the case where g_{jk} and ϕ_j are as given in Fig. 7-2. Since all the above operators are not smoothing, the Vidyasagar's digraph $G_V(CSF)$ has cycles and, so, his theorem cannot guarantee well-posedness. Now, let us apply our theorems. As this system is System I, we can apply Theorem 4-1. By Theorem 2-1 and Theorem 2-2, we obtain the upper bounds of the uniform instantaneous gain as given in Table 7-1. So, by the condition (i) of Theorem 4-1, the test matrix is calculated as

$$I - \tilde{B}\tilde{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 0.05 & 0.02 \\ 0.09 & 0.05 \end{pmatrix}$$

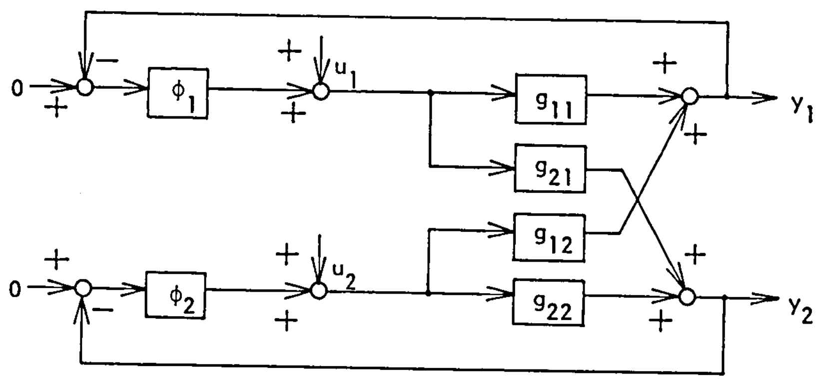

Fig. 7-1 System of Example 1

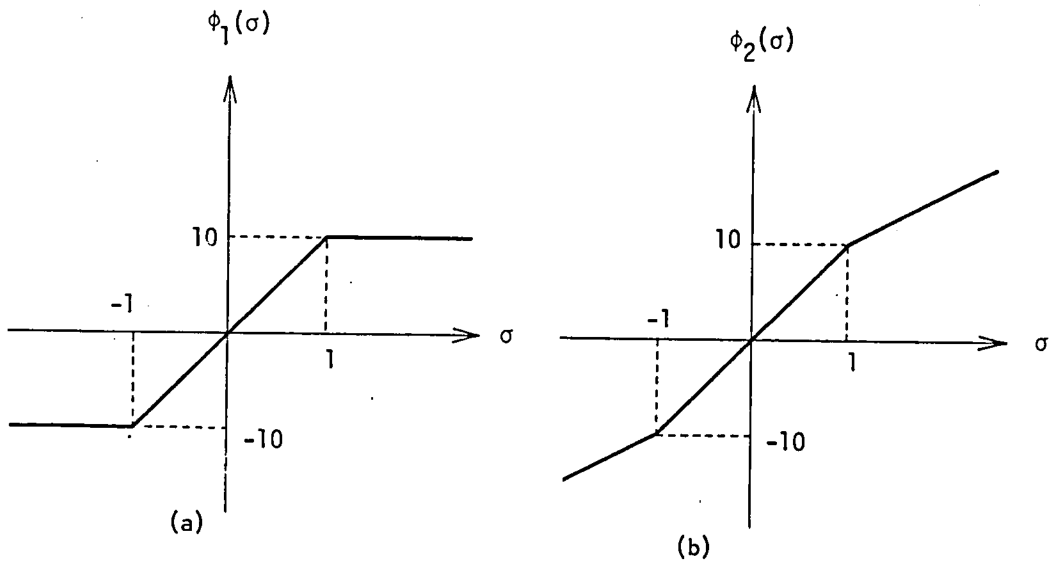

Fig. 7-2 Characteristics of ϕ_1 and ϕ_2

$$= \begin{bmatrix} 0.5 & -0.2 \\ -0.9 & 0.5 \end{bmatrix} \tag{7-1}$$

Since this matrix is an M-matrix, we can conclude that the system is well-posed in the sense of Definition 2-1. Next, let us apply the condition (ii) of Theorem 4-1. By (4-11), we obtain a = 0.109 and b = 10. As the loop-gain-product $a \cdot b = 1.09$ is greater than 1, the well-posedness of the system cannot be guaranteed by this condition. The main reason of the success of the condition (i) is the use of the weighted mean square norm.

Subsystem	Transfer function or characteristics	Uniform instantaneous gain
g_{11}	$\frac{1+0.5s}{(1+5s)(1+2s)} - 0.05$	0.05
g_{12}	$\frac{0.5}{(1+5s)(1+10s)} + 0.02$	0.02
g_{21}	$\frac{0.5(1+0.01s)}{(1+5s)(1+20s)} + 0.09$	0.09
g_{22}	$\frac{1}{(1+10s)(1+s)} + 0.05$	0.05
ϕ_1	given in Fig. 7-2 (a)	10
ϕ_2	given in Fig. 7-2 (b)	10

Table 7-1 Characteristics of the subsystems of Example 1 and their uniform instantaneous gains

Sec. 7.2. Example 2 (Application of Theorem 3-1 and Theorem 3-2 to a trasient stability analysis of a multi-machine power system)

Consider an n-machine power system which is described by

$$M_j \ddot{\delta}_j + D_j \dot{\delta}_j = P_{mj} - P_{ej} \qquad \text{for } j=1, \cdots, n \tag{7-2}$$

where δ_j is the absolute rotor angle of the j-th machine and P_{ej} is the electrical power delivered by the j-th machine which is given by

$$P_{ej} = \sum_{k=1}^{n} B_{jk} \cos(\delta_j - \delta_k - \theta_{jk}) \tag{7-3}$$

The other quantities are

M_j = inertia coefficient

D_j = damping coefficient

P_{mj} = mechanical power delivered to the j-th machine

$B_{jk} = E_j E_k Y_{jk}$

E_j = internal voltage

Y_{jk} = modulus of the transfer admittance between the j-th and k-th machines

θ_{jk} = phase angle of transfer admittance between the j-th and k-th machines

and all assumed to be constant. In addition, we assume uniform damping, namely,

$$\frac{D_j}{M_j} = \tilde{\lambda} \qquad \text{for } j=1, \cdots, n \tag{7-4}$$

We take the n-th machine as the comparison machine and, following Jocic et al. (1978), decompose the system into m = n-1 Lur'e-Postnikov type subsystems. As a result, we obtain the next equations:

$$\begin{aligned} \dot{\underline{x}}_j &= \tilde{A}_j \underline{x}_j + \underline{b}_j \{-\phi_j(y_j) + g_j(y_1, \cdots, y_m)\} \\ y_j &= \underline{c}_j^T \underline{x}_j \qquad \text{for } j=1, \cdots, m \end{aligned} \tag{7-5}$$

where

$\underline{x}_j = (\omega_{jn}, \delta_{jn} - \delta_{jn}^0)^T$

$\delta_{jk} = \delta_j - \delta_k$

δ_{jk}^0 = the steady state value of δ_{jk}

$$\tilde{A}_j = \begin{pmatrix} -\tilde{\lambda} & 0 \\ 1 & 0 \end{pmatrix} \qquad \underline{b}_j = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \underline{c}_j = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7-6}$$

$$\phi_j(y_j) = (M_j^{-1} + M_n^{-1}) B_{jn} \sin\theta_{jn} \cdot [\sin(y_j + \delta_{jn}^0) - \sin\delta_{jn}^0] \tag{7-7}$$

$$g_j(y_1, \cdots, y_m) = (M_n^{-1} - M_j^{-1}) B_{jn} \cos\theta_{jn} \cdot (\cos\delta_{jn} - \cos\delta_{jn}^0) + \sum_{\substack{k=1 \\ k\neq j}}^{m} (M_n^{-1} f_{nk} - M_j^{-1} f_{jk}) \tag{7-8}$$

$$f_{jk} = B_{jk}[\cos(\delta_{jk} - \theta_{jk}) - \cos(\delta_{jk}^0 - \theta_{jk})] \tag{7-9}$$

The nonlinear time-invariant function $\phi_j(y_j)$ given by (7-7) satisfies the next inequality in a finite region surrounding the equilibrium point $\underline{x}_j = \underline{0}$:

$$0 \leqq \zeta_j y_j^2 \leqq y_j \phi_j(y_j) \leqq \eta_j y_j^2 \qquad y_j \in [-\rho_{1j}, \rho_{2j}] \tag{7-10}$$

where $\eta_j = (M_j^{-1} + M_n^{-1}) B_{jn} \sin\theta_{jn}$, and $-\rho_{1j}$ and ρ_{2j} are the solutions of

$$\varepsilon_j y_j = \sin(y_j + \delta_{jn}^0) - \sin\delta_{jn}^0 \quad , \quad \varepsilon_j = \zeta_j / \eta_j \tag{7-11}$$

in the interval $(-\pi - 2\delta_{jn}^0, \pi - 2\delta_{jn}^0)$. The interconnection function $g_j(y_1, \cdots, y_m)$ given by (7-8) satisfies the next inequality:

$$g_j(y_1, \cdots, y_m) \leqq \beta_{j1}|y_1| + \cdots + \beta_{jm}|y_m| \tag{7-12}$$

where

$$\begin{aligned} \beta_{jj} &= |M_n^{-1} - M_j^{-1}| B_{jn} \cos\theta_{jn} + \sum_{k=1, k\neq j}^{n} M_j^{-1} B_{jk} \\ \beta_{jk} &= M_n^{-1} B_{kn} + M_j^{-1} B_{jk} \qquad (j \neq k) \end{aligned} \tag{7-13}$$

As for the details of the above formulation, refer to Jocic et al.†

Now, let us apply Theorem 3-2 to the above system. The system (7-5) is a special case of the system (3-3) where

$$\begin{aligned} \underline{\tilde{f}}_j(\underline{x}_j, t) &= \tilde{A}_j \underline{x}_j - \underline{b}_j \phi_j(\underline{c}_j^T \underline{x}_j) \\ \underline{\tilde{g}}_j(\underline{x}_j, t) &= \underline{b}_j g_j(\underline{c}_1^T \underline{x}_1, \cdots, \underline{c}_m^T \underline{x}_m) \end{aligned} \tag{7-14}$$

As a candidate for the Lyapunov function of each isolated subsystem, we adopt

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†) It should be noted that Jocic et al.'s paper (1978) includes a minor mistake in their equation (5.4). This mistake was caused by using the first inequality of (5.3) to evaluate the left-hand side of (2.13). The simplest way to correct this mistake is to replace (5.4) by $\xi_{ii} = \hat{\zeta}_i \eta_{ii}^{-1} [|M_n^{-1} - M_i^{-1}| A_{in} \cos\theta_{in} + \sum_k M_i^{-1} A_{ik}]$, $\xi_{ij} = \hat{\zeta} \eta_{ii}^{-1} [M_n^{-1} A_{jn} + M_i^{-1} A_{ij}]$ $(i \neq j)$, although a little more elaborate evaluation is possible. Because such a correction increases the value of ξ_{ij}, their method turns out to be unsuccessful in assuring stability for the example studied in their paper. However, this does not hurt the validity of their method at all and their result remains true if (5.4) is raplaced by the above.

$$v_j(\underline{x}_j) = \underline{x}_j^T P_j \underline{x}_j + 2\theta_j \int_0^{\underline{c}_j^T \underline{x}_j} \phi_j(\sigma) d\sigma \tag{7-15}$$

where P_j is a positive-definite matrix and θ_j is a non-negative constant. First, let us consider how the term $(\text{grad}_j v_j)^T \underline{\tilde{g}}_j(\underline{x}, t)$ can be bounded for this choice of $v_j(\underline{x}_j)$. Using (7-12) and $\underline{c}_j^T \underline{b}_j = 0$, we can easily obtain

$$(\text{grad}_j v_j)^T \underline{\tilde{g}}_j(\underline{x}, t) \leqq 2 |\underline{x}_j^T P_j \underline{b}_j| \sum_{k=1}^{m} \beta_{jk} |\underline{c}_k^T \underline{x}_k| \tag{7-16}$$

From this inequality, we are suggested to use

$$u_{1j} = |\underline{x}_j^T P_j \underline{b}_j| , \quad u_{2j} = |\underline{c}_j^T \underline{x}_j| \tag{7-17}$$

for this problem. The result on the absolute stability problem (Narendra and Taylor 1973) indicates that the inquality of the form (3-16) can be obtained by using P_j which satisfies the next three equations:

$$\tilde{A}_j^T P_j + P_j \tilde{A}_j = -\underline{q}_j \underline{q}_j^T \tag{7-18}$$

$$P_j \underline{b}_j - \underline{c}_j - \theta_j \tilde{A}_j^T \underline{c}_j = \underline{0} \tag{7-19}$$

$$\underline{q}_j = \kappa_j (\alpha_j \underline{c}_j + P_j \underline{b}_j) \tag{7-20}$$

where κ_j and α_j are constants. Since our subsystem is of order 2, the necessary and sufficient condition for the existence of the positive-definite solution P_j to the above set of equations can be written down as explicit inequalities on parameters, i.e.,

$$\alpha_j = -1, \ \tilde{\lambda}\theta_j \geqq 1, \ \tilde{\lambda}^2 \geqq 2\kappa_j^2 \tag{7-21}$$

When (7-21) is satisfied, the solution P_j is given by

$$P_j = \begin{pmatrix} \theta_j & 1 \\ 1 & \tilde{\lambda} \end{pmatrix} \qquad \theta_j = \frac{\tilde{\lambda} + (\tilde{\lambda}^2 - 2\kappa_j^2)^{1/2}}{\kappa_j^2} \tag{7-22}$$

From (7-14) and (7-15), we obtain

$$\dot{v}_j|_{(3-6)} = \underline{x}_j^T (P_j \tilde{A}_j + \tilde{A}_j^T P_j) \underline{x}_j - 2\phi_j (\underline{c}_j^T \underline{x}_j) \underline{c}_j^T \underline{x}_j \tag{7-23}$$

By substituting (7-18)-(7-20) and applying the inequality (7-10), we obtain

$$\dot{v}_j|_{(3-6)} = -\kappa_j^2 u_{1j}^2 + 2\kappa_j^2 u_{1j} u_{2j} - (\kappa_j^2 + 2\zeta_j) u_{2j}^2 \tag{7-24}$$

for $-\rho_{1j} \leqq \underline{c}_j^T \underline{x}_j \leqq \rho_{2j}$ (note $\underline{c}_j^T \underline{b}_j = 0$). Now, (7-16) and (7-24) correspond to (3-16) and (3-17) of Theorem 3-2, respectively, if we put $\kappa_j = \kappa_j^2$, $\lambda_j = \kappa_j^2$, $\mu_j = \kappa_j^2 + 2\zeta_j$, $u_{0j} = 0$, and $\gamma_{jk} = \beta_{jk}$. Here, from (7-17) and (7-21), we can easily verify that $u_{1j}(\underline{x}_j) = u_{2j}(\underline{x}_j) = 0$ is true only for $\underline{x}_j = \underline{0}$. For the above values of parameters, the test matrix $K^{1/2}M^{1/2} - \tilde{\Gamma}$ becomes

$$[K^{1/2}M^{1/2} - \tilde{\Gamma}]_{jj} = \kappa_j\sqrt{\kappa_j^2 + 2\zeta_j} - \kappa_j^2 - \beta_{jj}$$
$$[K^{1/2}M^{1/2} - \tilde{\Gamma}]_{jk} = -\beta_{jk} \qquad (j \neq k) \tag{7-25}$$

The M-matrix $K^{1/2}M^{1/2} - \Gamma$ is more likely to be an M-matrix as the diagonal elements become larger. Therefore, let us choose the value of κ_j which maximize the diagonal elements within the restriction (7-21). Then, we obtain $\kappa_j = \tilde{\lambda}/\sqrt{2}$ and $K^{1/2}M^{1/2} - \Gamma$ becomes $A = (a_{jk})$ where

$$a_{jj} = \tilde{\lambda}\sqrt{\tilde{\lambda}^2/4 + \zeta_j} - \tilde{\lambda}^2/2 - \beta_{jj}$$
$$a_{jk} = -\beta_{jk} \qquad (j \neq k) \tag{7-26}$$

Thus, we obtain the next result from Theorem 3-2.

Theorem 7-1

If A given by (7-26) is an M-matrix, the system (7-5) is asymptotically stable and $v(x) = \sum_{j=1}^{m} d_j v_j(x_j)$ becomes a Lyapunov function for some positive constants d_j's where

$$v_j(\underline{x}_j) = \underline{x}_j^T \begin{pmatrix} 2/\tilde{\lambda} & 1 \\ 1 & \tilde{\lambda} \end{pmatrix} \underline{x}_j + \frac{4}{\tilde{\lambda}} \int_0^{c_j^T \underline{x}_j} \phi_j(\sigma)d\sigma$$

Next, let us examine the relation to the result of (Jocic et al. 1978) and Theorem 3-1 (Araki 1978).

As for the power system problem studied above, Theorem 3-1 gives the next result.

Theorem 7-2

If $\tilde{\tilde{A}} = (\tilde{\tilde{a}}_{jk})$ given by

$$\tilde{\tilde{a}}_{jj} = \frac{\zeta_j}{\sqrt{\left(\frac{1+\zeta_j}{\tilde{\lambda}}\right)^2 + 1}} - \beta_{jj}$$

$$\tilde{\tilde{a}}_{jk} = -\beta_{jk}$$

is an M-matrix, the system (7-5) is asymptotically stable and $v(x) = \sum_{j=1}^{m} d_j v_j(\underline{x}_j)$ becomes a Lyapunov function for some positive constants d_j's where

$$v_j(\underline{x}_j) = \underline{x}_j^T \begin{pmatrix} (1+\zeta_j)/\tilde{\lambda} & 1 \\ 1 & \tilde{\lambda} \end{pmatrix} \underline{x}_j + 2\,\frac{1+\zeta_j}{\tilde{\lambda}} \int_0^{\underline{c}_j^T \underline{x}_j} \phi_j(\sigma)\,d\sigma$$

Let us compare Theorem 7-2 with Theorem 7-1. The values of the diagonal elements of the matrices A and $\tilde{\tilde{A}}$ are plotted for $\tilde{\lambda} = 0.5$ and $\beta_{jj} = 0$ in Fig. 7-3. As a matrix is more likely to be an M-matrix if the diagonal elements becomes larger, A is much more likely to be an M-matrix than $\tilde{\tilde{A}}$ if ζ_j are large. This means that Theorem 3-2 gives considerably better results than Theorem 3-1.

Next, let us compare Theorem 7-1 to Jocic et al.'s result. As they used a vector Lyapunov function method, from Sec. 3.1. we can expect that Theorem 7-1 is sharper than Jocic et al.'s.

To sum up, the sharpness of the stability conditions obtained from Theorem 7-1, Theorem 7-2, and Jocic et al.'s are expected to be in this order.

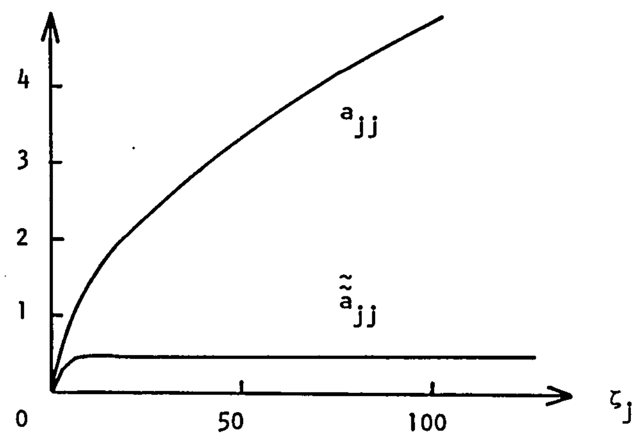

Fig. 7-3 Diagonal elements of the test matrices A and $\tilde{\tilde{A}}$ for $\tilde{\lambda}=0.5$ and $\beta_{jj}=0$.

		$\varepsilon = 0.99$	$\varepsilon = 0.5$
Case 1	Jocic et al's	cannot assure stability	cannot assure stability
	Theorem 7-2	$\tilde{\lambda} \geq 0.59$	$\tilde{\lambda} \geq 0.64$
	Theorem 7-1	$\tilde{\lambda} \geq 0.17$	$\tilde{\lambda} \geq 0.25$
Case 2	Jocic et al's	cannot assure stability	cannot assure stability
	Theorem 7-2	$\tilde{\lambda} \geq 24.0$	$\tilde{\lambda} \geq 27.0$
	Theorem 7-1	$\tilde{\lambda} \geq 2.6$	$\tilde{\lambda} \geq 4.5$

Table 7-2 Values of $\tilde{\lambda}$ for which the three methods can assure stability

Lastly, let us study a numerical example by applying Jocic et al's method, Theorem 7-2, and Theorem 7-1. Consider a three-machine system with

$$M_1 = 0.01,\ M_2 = 0.01,\ M_3 = 2.0$$

$$E_1 = E_2 = E_3 = 1.0$$

$$\delta^0_{12} = 5^0,\ \delta^0_{13} = 2^0,\ \delta^0_{23} = -3^0$$

$$\theta_{12} = 86^0,\ \theta_{13} = 88^0,\ \theta_{23} = 89^0$$

Let us study two cases:

Case 1) $Y_{12} = 1\times10^{-3}$, $Y_{13} = 0.1$, $Y_{23} = 0.1$

Case 2) $Y_{12} = 0.1$, $Y_{13} = 1.0$, $Y_{23} = 1.0$

Case 1 was studied by Jocic et al. (1978) before (concerning their result, refer to footnote in page 108). For the above system, the three methods can respectively assure stability for the values of the parameters $\tilde{\lambda}$ and ε ($= \zeta_1/\eta_1 = \zeta_2/\eta_2$) tabulated in Table 7-2.

Here, note that if the values of ε is taken smaller, the region $[-\rho_{1j}, \rho_{2j}]$ given by (7-10) becomes larger and so we can obtain a larger estimate of the stability region. Also note that smaller value of $\tilde{\lambda}$ means that the generator is less damped and that it is more difficult to assure stability. For these reasons if a method can assure stability for the system with smaller values of $\tilde{\lambda}$ and ε, the method is more powerful. Table 7-2 shows our Theorem 7-1 is considerably more powerful than the other two methods. The value of $\tilde{\lambda}$ often observed in actual power systems is about 0.05 and those of Y_{jk} are a little smaller than 1.

Sec. 7.3. Example 3 (Stability region estimates using Theorem 3-3 and Weissenberger's method)

Consider the system described by

$$\begin{aligned} \dot{\underline{x}}_1 &= A_1\underline{x}_1 + \underline{g}_1(\underline{x}, t) \\ \dot{x}_2 &= -6x_2 + 4x_2^3 + g_2(\underline{x}, t) \\ \dot{x}_3 &= -8x_3 + 5x_3^3 + g_3(\underline{x}, t) \end{aligned} \qquad (7\text{-}27)$$

in the region

$$||\underline{x}_1|| \leq 5, \ |x_2| \leq 1, \ |x_3| \leq 1 \tag{7-28}$$

where $\underline{x}_1$ is a 2-vector and

$$A_1 = \begin{pmatrix} -3.16 & 0 \\ 3.16 & -12.64 \end{pmatrix}$$

We assume that the interconnection functions $g_j(\underline{x}, t)$ satisfy

$$\begin{aligned} ||\underline{g}_1(\underline{x}, t)|| &\leq 2|x_2| + 3|x_3| \\ |g_2(\underline{x}, t)| &\leq 0.5||\underline{x}_1|| + |x_3| \\ |g_3(\underline{x}, t)| &\leq ||\underline{x}_1|| + |x_2| \end{aligned} \tag{7-29}$$

First, let us apply Theorem 3-2 and Theorem 3-3. For each subsystem we have a second order Lyapunov function

$$v_1 = \underline{x}_1^T P_1 \underline{x}_1, \ v_2 = 0.5(x_2)^2, \ v_3 = 0.5(x_3)^2 \tag{7-30}$$

where

$$P_1 = \begin{pmatrix} 0.5 & 0.024 \\ 0.024 & 0.12 \end{pmatrix}$$

Put $u_{11} = u_{12} = ||\underline{x}_1||$, $u_{21} = u_{22} = |x_2|$, $u_{31} = u_{32} = |x_3|$. Then the assumptions (a) and (b) are satisfied with $\kappa_1 = \mu_1 = 3$, $\lambda_1 = 0$, $\kappa_2 = \mu_2 = 2$, $\lambda_2 = 0$, $\kappa_3 = \mu_3 = 3$, $\lambda_3 = 0$, $\gamma_{11} = \gamma_{22} = \gamma_{33} = 0$, $\gamma_{12} = 2$, $\gamma_{13} = 3$, $\gamma_{21} = 0.5$, $\gamma_{23} = 1$, $\gamma_{31} = 1$, $\gamma_{32} = 1$ and $v_{01} = 2.9$, $v_{02} = v_{03} = 0.5$. So we obtain

$$M^{1/2}K^{1/2} - \tilde{\Gamma} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 3 \\ 0.5 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{7-31}$$

Let us apply the method of 3.3.2 to determine the values of d_j. From (3-40)

$$\Theta = \begin{pmatrix} 0 & 0.34 & 0.42 \\ 0.49 & 0 & 0.41 \\ 0.80 & 0.41 & 0 \end{pmatrix} \tag{7-32}$$

$I - \Theta$ is an irreducible M-matrix. Since $I - \Theta^T\Theta$ is not positive-definite, the vector $\underline{\pi} = (1/3, 1/3, 1/3)^T$ does not belong to the set P given by (3-41). Therefore, we obtain the set Q by Theorem 3-4. The minimum value of $||\underline{\pi}||$ in the set Q is given by $\underline{\pi}_0 = (0.41, 0.34, 0.26)^T$. The set P and the set Q are illustrated in Fig. 7-4, where P and Q are projected on the coordinate hyperplane $\pi_3 = 0$ and the inside of the dotted line and of the hexagons correspond to P and Q, respectively (P is determined by point by point calculation). In Fig. 7-4, π_0 is denoted by the circle. We obtain the estimate of the stability region as

$$\frac{\underline{x}_1^T P_1 \underline{x}_1}{(1.35)^2} + \frac{(x_2)^2}{(0.62)^2} + \frac{(x_3)^2}{(0.70)^2} \leqq 1 \qquad (7\text{-}33)$$

Next let us apply the method of Weissenberger. For the above problem we also have first-order Lyapunov functions $\tilde{v}_j$

$$\tilde{v}_1 = (\underline{x}_1^T P_1 \underline{x}_1)^{1/2}, \quad \tilde{v}_2 = |x_2|, \quad \tilde{v}_3 = |x_3| \qquad (7\text{-}34)$$

In the region (7-28), the coefficients of the comparison system become

$$A_1 = \begin{pmatrix} -3 & 2.92 & 4.39 \\ 1.42 & -2 & 1 \\ 2.92 & 1 & -3 \end{pmatrix} \qquad (7\text{-}35)$$

As the matrix A_1 does not satisfy Hick's condition, Weissenberger's method cannot be applied in this form. So, we restrict the region (7-28) as $\|\underline{x}_1\| \leqq 5$, $|x_2| < \alpha$ and $|x_3| < \alpha$ so that Hick's condition may be satisfied. The coefficient of the comparison system become

$$A_2 = \begin{pmatrix} -3 & 2.92 & 4.39 \\ 1.42 & -6+4\alpha^2 & 1 \\ 2.92 & 1 & -8+5\alpha^2 \end{pmatrix} \qquad (7\text{-}36)$$

The suprimum of α which makes A_2 satisfy Hick's condition is 0.393. So, let $\alpha = 0.39$, $v_{01} = 1.7$, $v_{02} = 0.39$ and $v_{03} = 0.39$. For this value of α, the vector $\underline{q}$ satisfying $A_2\underline{q} < \underline{0}$ is almost unique (with arbitrary scalar multiple) and given by $\underline{q} = (0.556, 0.193, 0.251)^T$. So, the estimate of the stability region by Weissenberger's method is obtained as

$$\max \left\{ \frac{(\underline{x}_1^T P_1 \underline{x}_1)^{1/2}}{0.86}, \frac{|x_2|}{0.3}, \frac{|x_3|}{0.39} \right\} \leqq 1 \qquad (7\text{-}37)$$

The estimates of the stability regions given by (7-33) and (7-37) are illustrated in Fig. 7-5. The volumes of the two estimates are 1.22 and 0.402, respectively, and the region given by (7-37) is contained in the region given by (7-33). Thus the estimate of our method is considerably larger than that obtained by Weissenberger's.

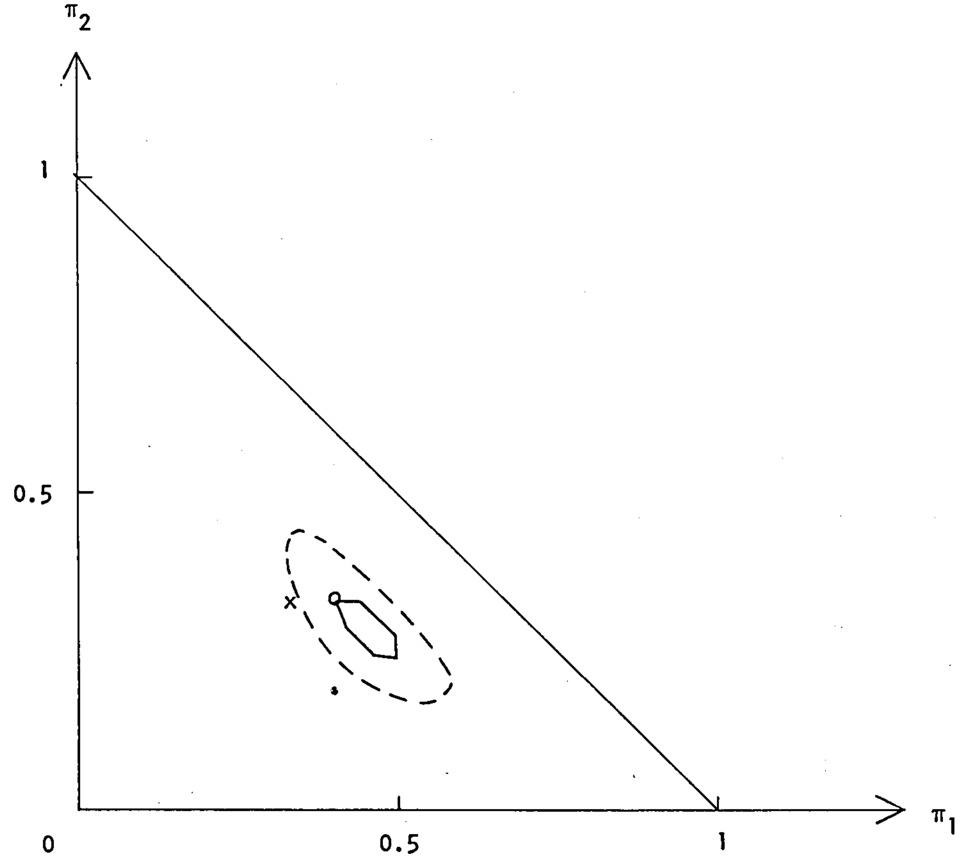

Fig. 7-4 Sets P and Q on π_3=0.

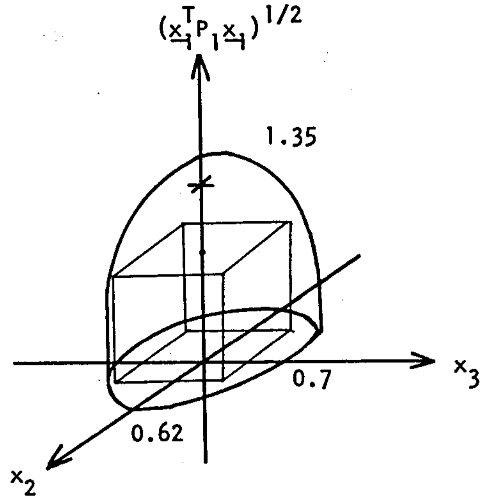

Fig. 7-5 Estimates of the stability region

Sec. 7.4. Example 4 (Application of the method of searching a feasible weight proposed in Sec. 4.4.)

Let us examine the method of searching a feasible weight proposed in Sec. 4.4. Let us study System I with

$$F(s) = \begin{pmatrix} \frac{1+0.6s}{(1+s)^3} & \frac{0.8}{(1+s)^2} & \frac{0.4}{(1+s)^2} \\ \frac{0.05+0.03s}{(1+s)^2} & \frac{1+0.6s}{(1+s)^3} & \frac{0.1+0.06s}{(1+s)^2} \\ \frac{0.1+0.06s}{(1+s)^2} & \frac{0.4}{(1+s)^2} & \frac{1+0.6s}{(1+s)^3} \end{pmatrix} \tag{7-38}$$

$$\zeta_1 = \zeta_2 = \zeta_3 = 0 \tag{7-39}$$

for the next two sets of η's.

Case a : $\eta_1 = \eta_2 = \eta_3 = 4.0$

Case b : $\eta_1 = \eta_2 = \eta_3 = 6.2$

As lower bounds ζ_j of nonlinearities ϕ_j are assumed to be zero, this example corresponds to Case 1 of System I.

Case a) Weighted multivariable circle criterion (Theorem 4-2) with $\tilde{W} = I$ cannot assure stability, because with this weight $Q(\omega)$ of (4-13) is not positive-semi-definite in the interval $0.8 \leq \omega \leq 1.8$. So, by applying the method of searching a feasible weight, we will choose another weight. In Step 1, $\tilde{\Gamma}(\omega)$ given by (4-59) is an M-matrix for $0.0 \leq \omega \leq 0.7$ and $2.0 \leq \omega$, and is not an M-matrix for $0.8 \leq \omega \leq 1.8$ (from this fact, Theorem 4-3 cannot assure stability). In Step 2, we choose 3 frequencies; $\omega_1=0.1$, $\omega_2=0.7$, $\omega_3=2.0$. In step 3, for these frequencies the diagonal matrices $\tilde{\tilde{W}}_{jk}(\omega_i)$ (j, k = 1, 2, 3) are calculated by (4-56) and (4-57). In Step 4, we obtain $\tilde{W}$ = (0.0587, 0.7336, 0.2150) as a candidate for a feasible weight. As the condition (4-13) is satisfied with this weight, this is found to be feasible. Namely, this example is guaranteed to be stable. To visualize the above procedure, the set $U(\omega_k)$ (k=1, 2, 3) and the weight $\tilde{W}$ are illustrated in Fig. 7-6, where they are projected on a coodinate plane $w_3=0$. In Fig. 7-6, three hexagons correspond to the set $U(\omega_k)$ (k=1, 2, 3) and the symbol x corresponds to $\tilde{W}$.

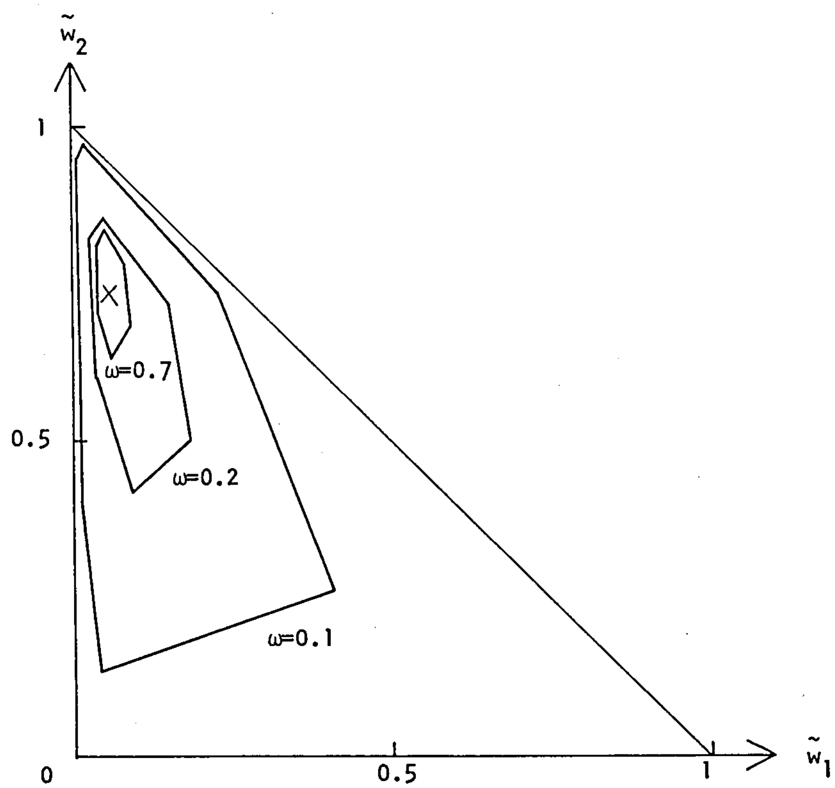

Fig. 7-6 The sets $U(\omega_i)$ and weight W on $\tilde{w}_3=0$ in Case a

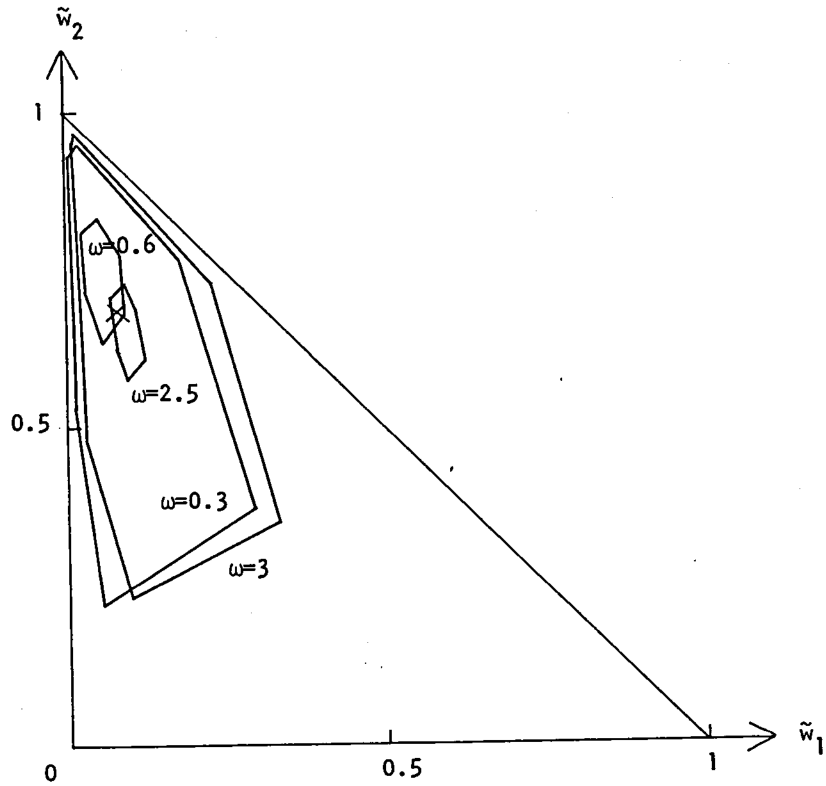

Fig. 7-7 The sets $U(\omega_i)$ and weight W on $\tilde{w}_3=0$ in Case b

Case b) As the values of η_j are greater than those of Case a, Theorem 4-2 cannot assure stability with $\tilde{W} = I$. Therefore, let us apply the method of searching a feasible weight. In step 1, $\tilde{\Gamma}(\omega)$ is an M-matrix in the interval $0.0 \leq \omega \leq 0.6$ and $2.5 \leq \omega$, and is not an M-matrix in the interval $0.7 \leq \omega \leq 2.0$. In Step 2, we choose four frequencies such as $\omega_1 = 0.3$, $\omega_2 = 0.6$, $\omega_3 = 2.5$, and $\omega_4 = 3.0$. We obtain $\tilde{W} = \mathrm{diag}(0.0743, 0.6857, 0.2437)$ as a candidate for a feasible weight. These results are illustrated in Fig. 7-7. Theorem 4-2 is satisfied with this weight $\tilde{W}$, and, so, this is found to be feasible.

Sec. 7.5. Example 5 (Comparison of the weighted multivariable circle criterion and other methods)

Let us study System I with

$$F(s) = \begin{pmatrix} \dfrac{1}{1 + s + s^2} & \dfrac{h_1}{1 + 2s + 3s^2} \\[2ex] \dfrac{h_2}{1 + 2s + s^2} & \dfrac{1}{1 + s + 3s^2} \end{pmatrix} \qquad (7\text{-}40)$$

$$\zeta_1 = \zeta_2 = 0, \quad \eta_1 = \eta_2 = 1$$

for various values of h_1 and h_2. The Rosenbrock's criterion using $D_4(s)$ (Theorem 4-4), the M-matrix condition (Theorem 4-3), and the weighted multivariable circle criterion together with the method of searching a feasible $\tilde{W}$ were applied. In Fig. 7-8 the regions of (h_1, h_2) in which stability was assured by those methods are drawn where a, b, c, and d correspond to

a: Stability region obtained by the Rosenbrock's criterion

b: Stability region obtained by the M-matrix condition

c: Stability region obtained by the weighted multivariable circle criterion with the method of searching a feasible $\tilde{W}$.

d: Hurwitz region

respectively. Here, by the term "Hurwitz region" we mean that the region in which the linear systems obtained by replacing $\phi_j(\underline{y}, t)$ with linear gains satisfying (4-5) are stable.

Next, System I with the same F(s) given by (7-40) was studied again. This time, ζ_j and h_j were fixed as

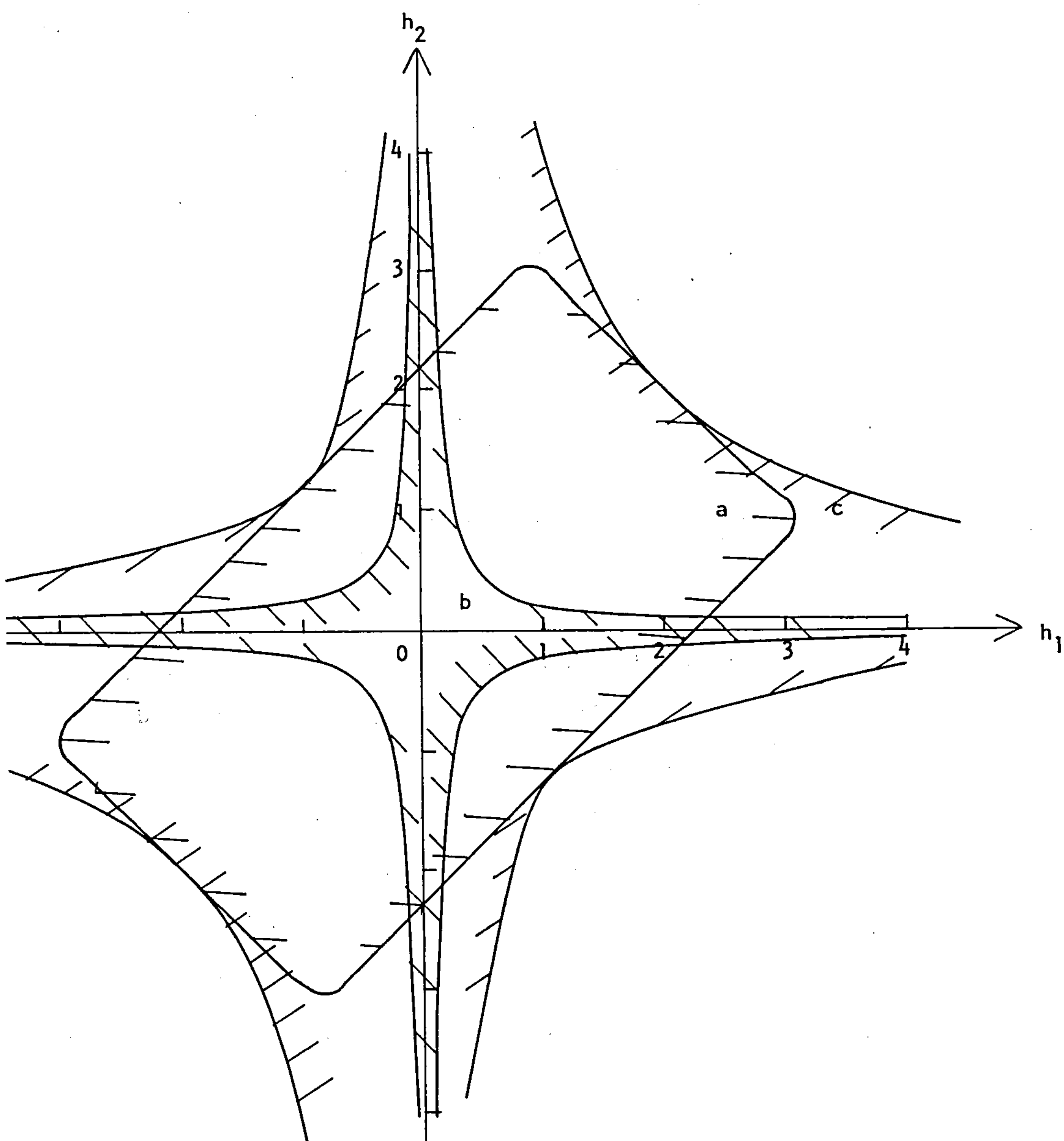

Fig. 7-8 Example 5 The region of d is given by the union of the region of c and the second and the fourth quadrants.

$$\zeta_1 = \zeta_2 = 0, \quad h_1 = 0.8, \quad h_2 = 3$$

and η_j's were changed. The region of (η_1, η_2) in which stability was assured by the three methods are drawn in Fig. 7-9.

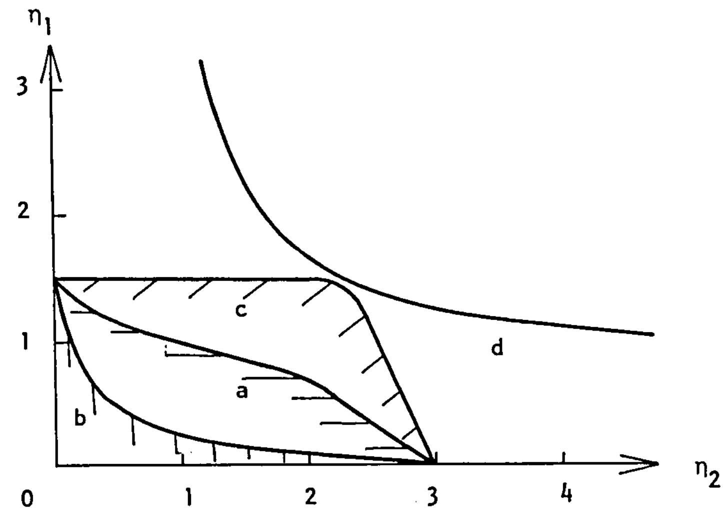

Fig. 7-9 Example 5

Sec. 7.6. Example 6 (Application of Theorem 5-2, Theorem 6-3, Theorem 6-7, and Weissenberger's method)

Let us study System IIA which is described by

$$\dot{\underline{x}}_j = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix} \underline{x}_j + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_j, \quad y_j = (1, 1)\underline{x}_j \quad j=1, 2, 3 \tag{7-41}$$

$$u_j = -41.6 \sin(y_j) + g_j(\underline{y}) \tag{7-42}$$

where the functions $g_j(\underline{y})$ are assumed to satisfy

$$|g_1(\underline{y})| \leqq \beta_{13}|y_3|, \ |g_2(\underline{y})| \leqq \beta_{21}|y_1|, \ |g_3(\underline{y})| \leqq \beta_{32}|y_2| \tag{7-43}$$

This system was treated by Šiljak (1978) before. Because of the form of the nonlinear functions, stability in the large cannot be expected for this system. In the following, let us compare our method with the previous one (Šiljak 1978) on local stability from two slightly different viewpoints.

First, let us fix the range of y_j as $|y_j| \leqq 1.94$ (j=1, 2, 3) so that the nonlinear function $\phi_j(y_j) = 41.6 \sin(y_j)$ satisfies the inequality

$$20\, y_j^2 \leqq \phi_j(y_j)y_j \leqq 41.6\, y_j^2 \qquad j=1, 2, 3 \tag{7-44}$$

Under this assumption, we calculate the value of the product $\beta_{13}\beta_{21}\beta_{32}$ for which each method can assure stability. First, let us apply the Šiljak's method. Define $\tilde{v}_j(\underline{x}_j)$ by

$$\tilde{v}_j(\underline{x}_j) = (\underline{x}_j^T H_j \underline{x}_j)^{1/2} \text{ where } H_j = \begin{pmatrix} 7.91 & 1.81 \\ 1.81 & 1.86 \end{pmatrix} \tag{7-45}$$

This function is a Lyapunov function of the isolated subsystem. According to the method given in Šiljak (1978), we obtain the result that the system is asymptotically stable for

$$\beta_{13}\beta_{21}\beta_{32} < (0.0417)^3 \tag{7-46}$$

Next, let us apply our method (Theorem 5-2). From the Nyquist locus of the transfer function of the linear part of (7-41), we can easily find that $\alpha_j \leqq 3.205$ makes the $h_j(s)$ of (5-33) positive-real. So, our Theorem 5-2 tells that the system is asymptotically stable if

$$\hat{A} - \hat{B} = \begin{bmatrix} 3.205 & 0 & 0 \\ 0 & 3.205 & 0 \\ 0 & 0 & 3.205 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \beta_{13} \\ \beta_{21} & 0 & 0 \\ 0 & \beta_{32} & 0 \end{bmatrix} \qquad (7\text{-}47)$$

is an M-matrix, i.e., if

$$\beta_{13}\beta_{21}\beta_{32} < (3.205)^3 \qquad (7\text{-}48)$$

is satisfied. Thus our method can allow considerably larger values for the interconnections.

Next, let us fix the values of these interconnections as $\beta_{13} = \beta_{21} = \beta_{32} = 0.0417$ and calculate estimates of the stability region by Weissenberger's method (1973) and by our method (Theorem 6-3 and Theorem 6-7). First, let us apply the former. From the Lyapunov functions of subsystems given by (7-45), we can obtain the following estimate of the stability region:

$$\tilde{S} = \{\underline{x} \mid \underline{x}_1^T H_1 \underline{x}_1 \leqq 2.64,\ \underline{x}_2^T H_2 \underline{x}_2 \leqq 2.64,\ \underline{x}_3^T H_3 \underline{x}_3 \leqq 2.64\} \qquad (7\text{-}49)$$

(For details of the method of calculation, refer to Weissenberger (1973)). Next, let us apply our method. For the above value of β_{13}, β_{21} and β_{32}, the M-matrix condition of Theorem 5-2 is satisfied if $\alpha_j > 0.0405$ (j=1, 2, 3). For α_j=0.042, the $h_j(s)$ of (5-32) is positive-real if $\zeta_j \geqq -3.199$. For this value of ζ_j (i.e., for ϕ_j to satisfy $-3.199\ y_j^2 \leqq \phi_j y_j \leqq 41.6\ y_j^2$), the admissible region of y_j is

$$-3.4 \leqq y_j \leqq 3.4$$

(The above bound is obtained by solving $(\sin y_j)/y_j = -3.199/41.6$). Now, set $\zeta_j = -3.199$, $\eta_j = 41.6$, $\alpha_j = 0.042$, $\varepsilon_j = 0.01$, and $\delta = 0.5\times10^{-3}$ and solve (6-23). Then, we obtain

$$P_j = \begin{bmatrix} 33.5 & 17.3 \\ 17.3 & 23.0 \end{bmatrix} \quad \text{for } j=1, 2, 3$$

(This is the maximum solution of the Riccati equation). Since $\rho_{1j} = \rho_{2j} = 3.4$, the v_{0j} of (6-38) becomes 249.58. For $\pi_1 = \pi_2 = \pi_3$, the matrix $\Pi - \Theta^T\Pi\Theta$ given by (3-40) becomes positive-definite. From the result of 3.3.2, we choose $d_1 = d_2 = d_3$ $(= \pi_j/v_{0j})$. Thus, we obtain the following estimate:

$$S = \{\underline{x} \mid \underline{x}_1^T P_1 \underline{x}_1 + \underline{x}_2^T P_2 \underline{x}_2 + \underline{x}_3^T P_3 \underline{x}_3 \leqq 249.58\} \qquad (7\text{-}50)$$

By a suitable transformation we can easily prove that S completely contains $\tilde{S}$. In order to visualize the difference of the two estimates, the intersections of S and $\tilde{S}$ with the hyperplane $\underline{x}_2 = \underline{x}_3 = \underline{0}$ are illustrated in Fig. 7-10.

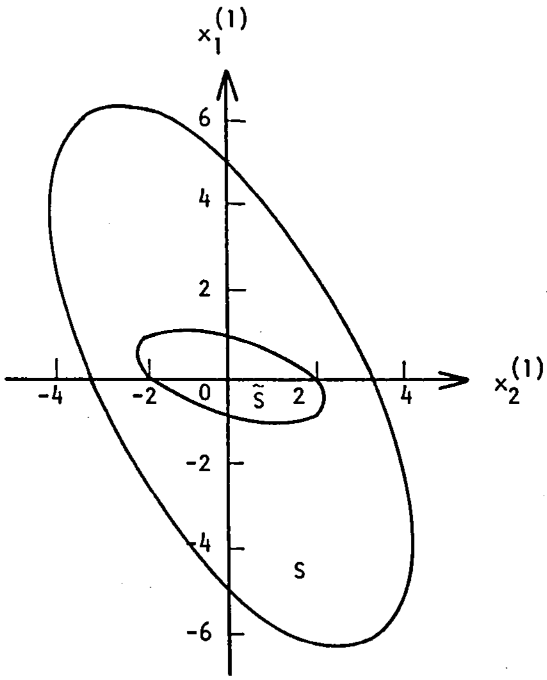

Fig. 7-10. Comparison of the two estimates of the stability region

Sec. 7.7. Example 7 (Further examination of Remark 5-15)

In Remark 5-15, we examined how to find the value of Θ_j which can be expected to satisfy the condition (5-112). The method proposed in Remark 5-5 ignors the influence of Θ_j on the off-diagonal elements of (5-112). To examine the influence, let us study the next example. In (5-112), we consider the case

$$F_j(s) = \begin{pmatrix} \dfrac{k_{11}}{1 + sT_{11}} & \dfrac{k_{12}}{1 + sT_{12}} \\ \dfrac{k_{21}}{1 + sT_{21}} & \dfrac{k_{22}}{1 + sT_{22}} \end{pmatrix} \tag{7-51}$$

$$\Theta_j = \text{diag}\ (\theta_1, \theta_2)\ , \quad \eta_j = \text{diag}\ (\eta_1, \eta_2) \tag{7-52}$$

where $k_{\ell\ell}$ and $T_{k\ell}$ $(k, \ell = 1, 2)$ are non-negative constants. In order to determine θ_{j1} and θ_{j2}, we use the next relation.

$$\min_{\omega} \mathrm{Re}\left\{\frac{(1 + i\omega\theta)k}{1 + i\omega T}\right\} = \begin{cases} k & \text{if } T \leqq \theta \\ \frac{\theta}{T} k & \text{if } T > \theta \end{cases} \tag{7-53}$$

$$\min_{\omega} \left|\frac{(1 + i\omega\theta)k}{1 + i\omega T}\right| = \begin{cases} \frac{\theta}{T} |k| & \text{if } T \leqq \theta \\ |k| & \text{if } T > \theta \end{cases} \tag{7-54}$$

We will choose the optimum values of θ_1 and θ_2 so that the Γ given by (5-113) becomes more likely to be an M-matrix (we examine M-matrix condition instead of (5-112)). For this example, we can easily find the best θ_1 and θ_2, and by using those values we obtain the M-matrix condition as shown in Table 7-3. This procedure is as follows. As we can choose θ_1 and θ_2 independently, we examine the optimum value only about θ_1. The leading principal minors of the matrix Γ_j become more likely to be positive if $\gamma_{kk}^{(j)}$ becomes larger and $\gamma_k^{(j)}$ becomes smaller. In order to have the largest value of $\gamma_{11}^{(j)}$, it is desirable to choose $T_{11} \leqq \theta_1$ by (7-53), and in order to have the smallest values of $\gamma_{12}^{(j)}$ it is desirable to choose $\theta_1 \leqq T_{12}$ by (7-54). In the first case of Table 7-3, as $T_{11} \leqq T_{12}$ is satisfied, T_1 can be chosen to satisfy these two conditions. But in the second case of Table 7-3, we cannot choose θ_1 to satisfy the both conditions. Here, if $\theta_1 \geqq T_{11}$, $\gamma_{11}^{(j)}$ does not change and $\gamma_{12}^{(j)}$ becomes larger as θ_1 becomes larger. And if $\theta_1 \leqq T_{12}$, $\gamma_{11}^{(j)}$ becomes smaller and $\gamma_{12}^{(j)}$ does not change as θ_1 becomes smaller. Therefore the optimum choice exists in the interval $T_{12} \leqq \theta_1 \leqq T_{11}$. If $T_{12} \leqq \theta_1 \leqq T_{11}$, the M-matrix condition becomes

$$\left(\frac{T_{12}}{T_{11}} k_{11} + \frac{T_{12}}{\theta_1} \eta_1^{-1}\right)(k_{22} + \eta_2^{-1}) > |k_{12}||k_{21}| \tag{7-55}$$

where θ_2 is chosen to satisfy $T_{22} \leqq \theta_2 \leqq T_{21}$. So the best choice of θ_1 is $\theta_1 = T_{12}$. The results of the other cases are obtained in the similar way.

According to Remark 5-15, we choose $\theta_1 = T_{11}$ and $\theta_2 = T_{22}$. From Table 7-3, this choice is valid when the time constants of off-diagonal elements are larger than those of diagonal elements.

		optimum θ_1, θ_2	M-matrix condition
Case 1	$T_{11} \leq T_{12}$ $T_{21} \geq T_{22}$	$T_{11} \leq \theta_1 \leq T_{12}$ $T_{22} \leq \theta_2 \leq T_{12}$	$(K_{11}+\eta_1^{-1})(K_{22}+\eta_2^{-1})-\|K_{21}\|\|K_{12}\| > 0$
Case 2	$T_{11} \geq T_{12}$ $T_{21} \geq T_{22}$	$\theta_1 = T_{12}$ $T_{22} \leq \theta_2 \leq T_{12}$	$(K_{11}+\frac{T_{11}}{T_{12}}\eta_1^{-1})(K_{22}+\eta_2^{-1})-\frac{T_{11}}{T_{12}}\|K_{21}\|\|K_{12}\| > 0$
Case 3	$T_{11} \leq T_{12}$ $T_{21} \leq T_{22}$	$T_{11} \leq \theta_1 \leq T_{12}$ $\theta_2 = T_{21}$	$(K_{11}+\eta_1^{-1})(K_{22}+\frac{T_{22}}{T_{21}}\eta_2^{-1})-\frac{T_{22}}{T_{21}}\|K_{21}\|\|K_{12}\| > 0$
Case 4	$T_{11} \geq T_{12}$ $T_{21} \leq T_{22}$	$\theta_1 = T_{12}$ $\theta_2 = T_{21}$	$(K_{11}+\frac{T_{11}}{T_{12}}\eta_1^{-1})(K_{22}+\frac{T_{22}}{T_{21}}\eta_2^{-1})-\frac{T_{11}T_{22}}{T_{12}T_{21}}\|K_{21}\|\|K_{12}\| > 0$

Table 7-3. Optimum value of $\Theta_j = \mathrm{diag}(\theta_1, \theta_2)$

Sec. 7.8. Example 8 (Application of Theorem 5-5, Theorem 6-5, Theorem 6-9, and Araki et al.'s method to the power system considered in Sec. 7.2)

Let us consider the multi-machine power system which was studied in Sec. 7.2. In Sec. 7.2, we have presented a method of applying Theorem 3-1 and Theorem 3-2. Unfortunately, this method cannot be applied to the composite system when a subsystem is of order greater than 2. This is because eqs. (7-18)-(7-20) are not solvable when the order of $\tilde{A}_j$ of (7-18) is greater than 2. In this section, we will apply Theorem 5-5 and the method of Araki et al. (1980). These methods are also derived by applying Theorem 3-2 and Theorem 3-1, respectively, but do not have the above restriction. We will compare the sharpness of these methods numerically.

As the matrix $\tilde{A}_j$ given by (7-6) contains zero eigenvalue, we cannot apply Theorem 5-5 to the system (7-5) directly. So, first, convert the composite system (7-14) by putting

$$\tilde{\phi}_j(y_j) = \phi_j(y_j) - \zeta_j y_j \tag{7-56}$$

in (7-14). Then, from (7-10) we obtain

$$0 \leq \tilde{\phi}_j(y_j)y_j \leq (\eta_j - \zeta_j)y_j^2 \tag{7-57}$$

and (7-14) is represented as

$$\begin{aligned} \dot{\underline{x}}_j &= A_j\underline{x}_j + \underline{b}_j\{ - \tilde{\phi}_j(y_j) + g_j(y_1, \cdots, y_m)\} \\ y_j &= \underline{c}_j^T\underline{x}_j \qquad \text{for } j=1, \cdots, m \end{aligned} \tag{7-58}$$

where

$$A_j = \tilde{A}_j - \underline{b}_j\zeta_j\underline{c}_j^T = \begin{pmatrix} -\tilde{\lambda} & -\zeta_j \\ 1 & 0 \end{pmatrix} \tag{7-59}$$

As this system (7-57) and (7-58) is System IIB, we can obtain the next stability criterion from Theorem 5-5 and Theorem 6-5.

Theorem 7-3

Let ζ_j be a positive number and let $f_j(s)$ be given by

$$f_j(s) = \frac{1}{s^2 + \tilde{\lambda}s + \zeta_j} \tag{7-60}$$

If there exist κ_j, μ_j, θ_j for each j such that

$$\frac{1}{\eta_j - \zeta_j} + \mathrm{Re}\{(1 + \theta_j\omega i)f_j(i\omega)\} - \frac{\kappa_j}{2}|(1 + \theta_j\omega i)f_j(i\omega)|^2$$
$$- \mu_j|\frac{\kappa_j}{\eta_j - \zeta_j} + \frac{1}{2}||f_j(i\omega)|^2 \geq 0 \qquad \text{for all } \omega \tag{7-61}$$

and if the mxm matrix $\tilde{\tilde{A}} - \tilde{\tilde{B}}$ is an M-matrix where

$$\tilde{\tilde{A}} = \mathrm{diag}(\sqrt{\kappa_j\mu_j}) \ , \quad \tilde{\tilde{B}} = (\beta_{jk}) \tag{7-62}$$

then there exists a positive-definite 2x2 matrix P_j which satisfies

$$\begin{aligned} &(\kappa_j + \frac{\eta_j - \zeta_j}{2})P_j\underline{b}_j\underline{b}_j^TP_j + P_j\{A_j - \frac{\eta_j - \zeta_j}{2}\underline{b}_j(\theta_j\underline{c}_j^TA_j + \underline{c}_j)\} \\ &+ \{A_j - \frac{\eta_j - \zeta_j}{2}\underline{b}_j(\theta_j\underline{c}_j^TA_j + \underline{c}_j)\}^TP_j + \mu_j\underline{c}_j\underline{c}_j^T = 0 \end{aligned} \tag{7-63}$$

$$\text{for } j=1, \cdots, m$$

there exists a diagonal matrix D with $d_j>0$ which makes $\hat{M}D\hat{M} - \hat{\Gamma}^TD\hat{\Gamma}$ positive-semi-definite where $\hat{M}$ and $\hat{\Gamma}$ is given by (6-31), and the function $v(\underline{x})$ defined by

$$v(\underline{x}) = \sum_{j=1}^{m} d_j\{ \underline{x}_j^TP_j\underline{x}_j + 2\theta_j\int_0^{\underline{c}_j^T\underline{x}_j} \tilde{\phi}_j(\sigma)d\sigma\} \tag{7-64}$$

is a Lyapunov function of (7-58).

We can obtain the estimate of the stability region by Theorem 6-9.

Araki et al. (1980) applied Theorem 3-1 to a multi-machine power system whose subsystems are multi-input multi-output. Following their method, we can obtain the next theorem for a single-input single-output case.

Theorem 7-4

If there exist σ_j, θ_j for each j such that

$$\frac{1}{\eta_j - \zeta_j} + \mathrm{Re}\{(1 + \theta_j \omega i) f_j(i\omega - \sigma_j)\} \geqq 0 \quad \text{for all } \omega \tag{7-65}$$

and if the mxm matrix $E = (e_{jk})$

$$e_{jj} = 2\sigma_j/(q_j h_j) - \beta_{jj} \tag{7-66}$$

$$e_{jk} = - \beta_{jk} \tag{7-67}$$

is an M-matrix where

$$q_j = 2\,\lambda_{max}^{1/2}\,[P_j^{1/2} \underline{b}_j \underline{b}_j^T P_j^{1/2}] \tag{7-68}$$

$$h_j = \lambda_{max}^{1/2}\,[P_j^{-1/2} \underline{c}_j \underline{c}_j^T P_j^{-1/2}]$$

and P_j is the solution of the next equation.

$$\frac{1}{2}(\eta_j - \zeta_j) P_j \underline{b}_j \underline{b}_j^T P_j + P_j [\hat{A}_j - \frac{1}{2}(\eta_j - \zeta_j)(\underline{b}_j \underline{c}_j^T + \theta_j \underline{b}_j \underline{c}_j^T \hat{A}_j)]$$

$$+ [\hat{A}_j - \frac{1}{2}(\eta_j - \zeta_j)(\underline{b}_j \underline{c}_j^T + \theta_j \underline{b}_j \underline{c}_j^T \hat{A}_j)]^T P_j$$

$$+ \frac{1}{2}(\eta_j - \zeta_j)(\hat{A}_j^T \underline{c}_j \theta_j + \underline{c}_j)(\hat{A}_j^T \underline{c}_j \theta_j + \underline{c}_j)^T = 0 \tag{7-69}$$

$$\hat{A}_j = A_j + \sigma_j I$$

then the function defined by (7-64) is a Lyapunov function of (7-58) where P_j is given by (7-69).

The most important difference between the above two theorems is that in Theorem 7-4 we need to solve a Riccati equation (7-69) to assure the existence of a Lyapunov function whereas in Theorem 7-3 we do not. By this difference, Theorem 7-3 is much easier to be applied than Theorem 7-4.

Now, let us examine the same numerical example that was examined in Example 2. We only consider Case 1. Theorem 7-3 and Theorem 7-4 can assure stability for those values of the parameters $\tilde{\lambda}$ and ε tabulated in Table 7-4. From Table 7-2 and Table 7-4, the sharpness of the stability conditions obtained from Theorem 7-1, Theorem 7-2, Theorem 7-3, Theorem 7-4 and Jocic et al.'s condition is expected to be in the next order:

Theorem 7-1 $\tilde{=}$ Theorem 7-3 > Theorem 7-2 > Theorem 7-4 > Jocic et al.'s condition

The numerical values which were obtained by applying Theorem 7-3 to this example for $\varepsilon = 0.507$ are tabulated in Table 7-5. From Table 7-5 (c) and (d), the estimate of the stability region is given by

$$S_x = \{\underline{x} \mid v(\underline{x}) \leqq 100\} \qquad (7\text{-}70)$$

	$\varepsilon = 0.99$	$\varepsilon = 0.507$
Theorem 7-3	$\tilde{\lambda}_1 \geqq 0.17,\ \tilde{\lambda}_2 \geqq 0.185$ $\begin{pmatrix} \theta_1 = \theta_2 = 10 \\ \kappa_1 = \kappa_2 = 0.02 \\ \mu_1 = \mu_2 = 18 \end{pmatrix}$	$\tilde{\lambda}_1 \geqq 0.25,\ \tilde{\lambda}_2 \geqq 0.26$ $\begin{pmatrix} \theta_1 = \theta_2 = 8 \\ \kappa_1 = \kappa_2 = 0.033 \\ \mu_1 = \mu_2 = 10.9 \end{pmatrix}$
Theorem 7-4	$\tilde{\lambda}_1 \geqq 4.8,\ \tilde{\lambda}_2 \geqq 5.1$ $\begin{pmatrix} \theta_1 = \theta_2 = 10 \\ \sigma_1 = \sigma_2 = 1.1 \end{pmatrix}$	cannot assure stability

Table 7-4. Values of $\tilde{\lambda}_j$ for which Theorem 7-3 and theorem 7-4 can assure stability

β_{11}	β_{12}	β_{21}	β_{22}
0.516	0.155	0.159	0.291

(a) Interconnection parameters

	η_j	ζ_j	ε_j	ρ_{1j}	ρ_{2j}
Subsystem 1	12.03	6.1	0.507	1.936	1.823
Subsystem 2	11.05	5.6	0.507	1.797	1.965

(b) Subsystem parameters

	θ_j	$\tilde{\lambda}_j$	κ_j	μ_j	$\sqrt{\kappa_j\mu_j}$
Subsystem 1	8	0.25	0.033	10.91	0.6
Subsystem 2	8	0.26	0.033	10.91	0.6

(c) Frequency domain test (7-61)

	p_{11}	$p_{12}=p_{21}$	p_{22}
Subsystem 1	0.028	0.689	17.17
Subsystem 2	0.0336	0.760	17.40

(d) Solution of Riccati equation (7-63)

	v_{0j}	d_j
Subsystem 1	64.5	0.775
Subsystem 2	73.1	0.685

(e) Parameters for stability region estimate

Table 7-5. Parameter values for $\varepsilon = 0.507$

Chapter 8 Conclusion

Let us summarize the main results obtained in the preceding chapters.

In Chapter 2, we presented a new well-posedness condition for large-scale systems (Theorem 2-3). The condition can be tested easily if the uniform instantaneous gain (or its upper bound) of each operator, which corresponds to a subsystem, is known. For linear systems and memoryless systems, Theorem 2-1 and Theorem 2-2 give upper bounds of the uniform instantaneous gains. The condition can be interpreted in terms of a digraph derived from the system equation. Theorem 2-3 includes Vidyasagar's result (1980) as a special case. If we consider the simplest structure, our condition reduces to Willems's result (1971) for single-loop systems.

In Chapter 3, we presented a new stability criterion (Theorem 3-2) which is applicable to a wide class of composite systems. This criterion is obtained on condition that each subsystem has a second order Lyapunov function in a finite region surrounding the origin. This theorem deeply relates to Araki's criterion (Theorem 3-1) and it was shown that Theorem 3-2 includes Theorem 3-1 as a special case. This fact was examined numerically in Example 2 of Chapter 7 by applying these theorems to a transient stability analysis of multi-machine power systems. When Theorem 3-2 can assure stability, there exists a stability region, and Theorem 3-3 gives the estimate of this region. This theorem was compared with the result previously obtained by Weissenberger in Example 3 of Chapter 7, and Theorem 3-3 gives a considerably larger estimate than Weissenberger's method.

In Chapter 4 and Sec. 6.1, we studied a system illustrated in Fig. 4-1 in detail. First, the well-posedness condition (Theorem 4-1) of the system was obtained by using Theorem 2-1 - Theorem 2-3. Next, we obtained the weighted multivariable circle criterion (Theorem 4-2) and the M-matrix condition (Theorem 4-3). These conditions are represented in a frequency domain term. It was shown that if the M-matrix condition is satisfied, there exists a weight such that the weighted multivariable circle criterion is satisfied and that the M-matrix condition agrees with the L_2-stability condition previously obtained by Araki. The weighted multivariable circle criterion was compared with

Rosenbrock's four criteria, and it was shown that they are obtained from the weighted multivariable circle criterion by choosing the value of the weight appropriately. In other words, there exists a weight such that the weighted multivariable circle criterion is sharper than both M-matrix condition and Rosenbrock's four criteria. In Sec. 4.4, we proposed a simple method to search such a feasible weight. In Examples 4 and 5 of Chapter 7, the weighted multivariable circle criterion with the method of searching a feasible weight gives sharper results than the other two conditions. In Chapter 6, we presented a method of constructing a Lyapunov function (Theorem 6-1) by using the result of the weighted multivariable circle criterion, and we gave a method of estimating the stability region by using the Lyapunov function on condition that the system description is valid only for finite values of the output.

In Chapter 5 and Sec. 6.2, we studied composite systems illustrated in Fig. 5-1 and Fig. 5-2 in detail. First, it was shown that the composite systems are well-posed (Theorem 5-1). In Sec. 5.2, we presented a circle criterion type condition for the system of Fig. 5-1 (a)(b) and extended this criterion to the system of Fig. 5-2 (a)(b) (Theorem 5-2 and Theorem 5-3). In Sec. 5.3, we presented a Popov criterion type condition for the system of Fig. 5-1 (a)(c) and extended this criterion to the system of Fig. 5-2 (a)(b) (Theorem 5-5 and Theorem 5-6). The above four theorems are represented in a frequency domain term, namely, they do not contain the parameters of Lyapunov functions used in the proofs of these theorems. They deeply relate to L_2-stability conditions obtained by Araki (1976) and Saeki et al. (1978). It was shown that Theorem 5-2 can assure asymptotic stability in the large under weaker assumptions than the L_2-stability condition of Araki, and it was shown that Theorem 5-5 is sharper than the L_2-stability condition of Saeki et al. and implies Theorem 5-2 as a special case. In Sec. 6.2, we presented a method of constructing Lyapunov functions from the results of Theorems 5-2, 5-3, 5-5 and 5-6 (Theorems 6-3 - 6-6, respectively), and gave a concrete way of calculating an estimate of the stability region by using those Lyapunov functions (Theorems 6-7 - 6-10, respectively). In the analysis of composite systems, construction of the Lyapunov functions of subsystems has been the bottleneck in the overall procedure. The results obtained in

Chapters 5 and 6 carry both "handiness" and "goodness" to certain extent. We can test the Lyapunov stability of composite systems in frequency domain, which enables us to obtain a comparatively transparent view over the influence of the parameters on the overall stability. We can obtain subsystem Lyapunov function systematically by using these theorems. In Example 6 of Chapter 7, Theorem 5-2, Theorem 6-3 and Theorem 6-7 were applied to a simple system, and these results were compared with the result obtained by Weissenberger. In Example 8 of Chapter 7, Theorem 5-5 was applied to the multi-machine power system which was treated in Example 2.

Appendix M-Matrices

Fundamental properties of M-matrices are enumerated. The details are left to Fiedler & Pták (1962), Araki (1974, 1976), and Berman & Plemmos (1979).

Definition A-1

A square matrix $\Gamma = (\gamma_{jk})$ is said to be an *M-matrix* if and only if the off-diagonal elements are all non-positive and the leading principal minors of Γ are all positive.

Lemma A-1

Let Γ be a real square matrix with non-positive off-diagonal elements. Then the next four conditions are mutually equivalent.

(i) The leading principal minors of Γ are all positive.

(ii) There is a vector $\underline{x} = (x_1, \cdots, x_m)^T$ with $x_j > 0$ such that the elements of $\Gamma\underline{x}$ are all positive.

(iii) There is a vector $\underline{\tilde{x}} = (\tilde{x}_1, \cdots, \tilde{x}_m)^T$ with $\tilde{x}_j > 0$ such that the elements of $\Gamma^T\underline{\tilde{x}}$ are all positive.

(iv) Γ is nonsingular and its inverse $\Gamma^{-1} = (\hat{\gamma}_{jk})$ satisfies $\hat{\gamma}_{jj} > 0$ and $\hat{\gamma}_{jk} \geqq 0$ $(j \neq k)$.

Lemma A-2

If $\Gamma = (\gamma_{jk})$ is an M-matrix, any $\tilde{\Gamma} = (\tilde{\gamma}_{jk})$ that satisfies $\tilde{\gamma}_{jj} \geqq \gamma_{jj}$ for all j and $0 \geqq \tilde{\gamma}_{jk} \geqq \gamma_{jk}$ for all j, k $(j \neq k)$ is an M-matrix.

Lemma A-3

If Γ is an M-matrix and $D = \mathrm{diag}(d_j)$ with $d_j > 0$, ΓD and $D\Gamma$ are M-matrices.

Lemma A-4

Let R be a positive-definite diagonal matrix. Let $\Delta = (\delta_{jk})$ be a non-negative matrix. If $R - \Delta$ is an M-matrix, there is a positive-definite diagonal matrix W such that the matrix

$$Q = RWR - \Delta^T W \Delta$$

is positive-definite. In addition, for the W, the matrix

$$\tilde{Q} = RWR - \tilde{\Delta}^T W \tilde{\Delta}$$

is positive-definite for any $\tilde{\Delta} = (\tilde{\delta}_{jk})$ satisfying

$$\delta_{jk} \geqq |\tilde{\delta}_{jk}|$$

Lemma A-5 (Varga 1962)

Let Γ be an irreducible M-matrix, all the elements of Γ^{-1} are positive.

There are many other properties. Some of them are enumerated in Sec. 4.4.

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